Shiftable Intervals

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Abstract: Let a set of n fixed length intervals and a set of n (larger) windows, in one-to-one correspondence with the intervals, be given, and assume that each interval can be placed in any position within its window. If the position of each interval has been fixed, the intersection graph of such set of intervals is an interval graph. By varying the position of each interval in all possible ways, we get a family of interval graphs. In the paper we define some optimization problems related to the clique, stability, chromatic, clique cover numbers and cardinality of the minimum dominating set of the interval graphs in the family, mainly focussing on complexity aspects, bounds and solution algorithms. Some problems are proved to be NP-hard, others are solved in polynomial time on some particular classes of instances, which are characterized in the paper. Many practical applications can be reduced to these kind of problems, suggesting the use of Shiftable Intervals as a new interesting modeling framework.

Keywords: Interval Graphs, Optimization problems, Complexity

1. Introduction and general definitions

Interval graphs are a well studied modelling framework (Gilmore and Hoffman 1964, Golumbic 1980). Many practical applications can be naturally approached by using interval graphs; among the others, we recall applications in the field of biology, chemistry, archeology, project management, and scheduling. In all such cases, the objects involved in the problem are usually represented as fixed end intervals of a given axis (temperature, time, …). A reasonable extension is to introduce a flexibility in the definition of the intervals, that is, to allow each interval to move within a larger interval (*window*). By varying the position of each interval in all possible ways within its window, we get a family of interval graphs. In this paper we study this generalized modeling framework. We shall define some optimization problems related to the clique, stability, chromatic, clique cover numbers and cardinality of the minimum dominating set of the interval graphs in the family, mainly focussing on complexity aspects, bounds and solution algorithms.

Let T denote a set of n triples $t_i = <l_i,r_i,\lambda_i>$ of non-negative integer numbers satisfying $r_i - l_i \geq \lambda_i > 0$, i.e. $T = \{t_i = <l_i,r_i,\lambda_i> \in Z^3_+ : r_i - l_i \geq \lambda_i > 0$, for $i = 1,\ldots,n\}$. The interval $(l_i,r_i]$, open on the left, will be called *window* w_i (l_i,r_i will be called the *left* and *right endpoint* of the window, respectively), and the value λ_i will be called the *length of the interval* associated

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with window w_i. Unless stated otherwise, all intervals and windows considered in the paper are open on the left.

The set T can be thought of as a set of intervals open on the left and of prescribed length, each of which is free to move within the corresponding window. In other words, both endpoints of the i^{th} interval have to lay within the range $(l_i,r_i]$. The distance φ_i between the left endpoint of the i^{th} interval and the left endpoint l_i, of the corresponding window $w_i=(l_i,r_i]$ can be used for describing the exact position of the interval, and precisely, $l_i+\varphi_i$ ($l_i+\varphi_i+\lambda_i$, respectively) represents the coordinate of the left (right, resp.) endpoint of interval i (notice that $l_i+\varphi_i \leq l_i+\varphi_i+\lambda_i$, as $\lambda_i>0$). In what follows, the notation $<t_i,\varphi_i>$ will represent the i^{th} interval placed according to φ_i, that is, it will represent the interval $(l_i+\varphi_i, l_i+\varphi_i+\lambda_i]$.

The vector $\varphi = [\varphi_1, \varphi_2,\ldots, \varphi_n]$ of all φ_i's is called *placement*. A placement φ is *feasible* if $l_i \leq l_i+\varphi_i$ and $l_i+\varphi_i+\lambda_i, \leq r_i$, that is if $\varphi_i \in [0, r_i-l_i-\lambda_i]$ for all i (in the sequel, we shall deal with feasible placements only). Since $l_i,r_i,\lambda_i \in Z_+$, without loss of generality we shall limit ourselves to considering integer valued φ's only. Note that every φ_i may assume (at least) one out of a certain number of consecutive non–negative integers (the first one being zero).

The *interval model* $M(\varphi) = \{<t_1,\varphi_1>, <t_2,\varphi_2>,\ldots, <t_n,\varphi_n>\}$ is the set of all intervals placed according to φ. The intersection graph $G(\varphi)$ of this set of intervals of the real line is an interval graph (we say that two intervals $(a,b]$ and $(a',b']$ intersect when $a' < b \leq b'$).

The set of all interval graphs $G(\varphi)$ obtained by varying φ in all possible (feasible) ways is called the *family* $\mathcal{F}_T$ associated with the given triple set T. Notice that different values of the placement vector φ, hence different interval models, may give rise to the same interval graph, and notice also that for any graph $G \in \mathcal{F}_T$ there exists a feasible placement φ such that $G(\varphi)$ is isomorphic to G. Notice also that both $M(\varphi)$ and $\mathcal{F}_T$ are non-empty sets, since $\varphi=0$ is a placement which is always feasible.

A minimization (maximization, respectively) problem on a triple set T is defined as follows:

Problem Min $f(T)$ (Max $f(T)$, respectively):

Given: a triple set $T \equiv \{t_i=<l_i,r_i,\lambda_i> \in Z_+^3\colon r_i-l_i \geq \lambda_i > 0, \text{ for } i=1,\ldots,n\}$
and a function $f\colon \mathcal{F}_T \rightarrow Z_+$,

Find: a graph $G \in \mathcal{F}_T$,

Such That: $f(G)$ is minimum (maximum, resp.) over all graphs in $\mathcal{F}_T$.

In other words, an optimization problem on T consists of identifying an interval graph $G \in \mathcal{F}_T$ on which $f(G)$ attains its optimum value. $f(T)_{min}$ ($f(T)_{max}$, respectively) will denote the optimum value resulting from a mimimization (maximization) problem on T.

In this paper we shall consider the minimization and maximization problems on T arising when f is a function which associates to a graph $G\in\mathcal{F}_T$ some classical measures on graphs, and precisely the clique number $\omega(G)$ (i.e. the size of a maximum sized complete subgraph of G), the chromatic number $\chi(G)$ (i.e. the size of a minimum sized node coloring of G), the stability number $\alpha(G)$ (i.e. the size of a maximum sized independent set of G), the clique cover number $k(G)$ (i.e. the size of a minimum sized covering of G by complete subgraphs), and the size of the minimum dominating set $d(G)$. Notice that since, in this paper, we define f to be an optimization function itself, an optimization problem on T, in practice, is either one of the following four types of problem: min–min, min–max, max–min, max–max (for example, problem Min $\omega(T)$ is a min–max type problem, in fact a graph $G\in\mathcal{F}_T$ has to be found, with a <u>min</u>imum sized <u>max</u>imum complete subgraph).

From time to time we shall also refer to the corresponding decision version of the optimization problems described above. Its formal statement is the following:

Problem Min $f(T)$ (Max $f(T)$, respectively) in decision form:

Given:	a triple set $T \equiv \{t_i=<l_i,r_i,\lambda_i> \in Z_+^3: r_i-l_i \geq \lambda_i > 0$, for $i=1,\ldots,n\}$, a function f: $\mathcal{F}_T \rightarrow Z_+$, and a positive integer h,
Find:	a graph $G\in\mathcal{F}_T$,
Such That:	$f(G) \leq h$ ($f(G) \geq h$, resp.).

As discussed above, for any graph $G\in\mathcal{F}_T$ there exists a feasible placement φ such that $G(\varphi)$ is isomorphic to G. Thus, in all problem statements we can replace "Find a graph $G\in\mathcal{F}_T$, Such That $f(G)$…" with "Find a feasible placement $\varphi \in Z_+^n$, Such That $f(G(\varphi))$…".

For some of the defined problems we shall devise polynomial algorithms. Other ones will be proved to be NP-hard, and we shall prove some lower and upper bounds for them. Finally, we shall try to characterize sets of triples for which the problems that are difficult in the general case, can be solved in polynomial time. It is worth noticing that the complexity of solving the optimization problems defined above lays in the fact that we have to identify an interval graph of the family which achieves the optimum over the chosen objective function, and not on the complexity of evaluating such function on an interval graph; in fact, computing $\omega(G)$, $\chi(G)$, $\alpha(G)$, $k(G)$, $d(G)$ on a given interval graph G with n nodes takes $O(n \log n)$ time (Gupta et al. 1982, Farber 1982).

Many practical applications can be reduced to these kinds of problems on Shiftable Intervals. Take as an example some problems in the context of scheduling where jobs with ready and due dates are to be scheduled on a set of identical machines: the ready and due dates of a job can be seen as the left and right endpoint of a window, respectively, and its processing time as the interval length associated to the same window.

The paper is organized as follows. In Section 2 some definitions of particular sets of triples and some basic properties are introduced. Then it considers the problems related to the clique and the chromatic numbers (Section 3), the stability and the clique cover numbers (Section 4), and the cardinality of a dominating set (Section 5). A final section contains some concluding remarks and some directions for future work.

2. Definitions and basic properties

Let a set $T \equiv \{t_i{=}{<}l_i,r_i,\lambda_i{>} \in Z^3_+ : r_i{-}l_i \geq \lambda_i > 0$, for $i = 1,\ldots,n\}$ of triples be given. Consider the intersection graph $H_T = (V,E)$ of the set of windows $\{(l_i,r_i]: i{=}1,\ldots,n\}$ corresponding to triples in T where the nodes of V are in one–to–one correspondence with the windows, and an edge connects two nodes u, v if and only if the corresponding windows intersect.

Consider any two intersecting windows $w_u{=}(l_u,r_u]$ and $w_v{=}(l_v,r_v]$ corresponding to triples $t_u,t_v{\in}T$, and let (u,v) be the corresponding edge of H_T. Among the edges of H_T, we distinguish two types:

Definition 2.1: An edge $(u,v){\in}E$ is *strong* if and only if $l_u{+}\lambda_u \geq r_v{-}\lambda_v$ holds when $l_u \leq l_v \leq r_u \leq r_v$, or both $l_u{+}\lambda_u \geq r_v{-}\lambda_v$ and $l_v{+}\lambda_v \geq r_u{-}\lambda_u$ hold when $l_u \leq l_v < r_v \leq r_u$. An edge is *weak* in all other cases.

The set of all strong edges of H_T will be denoted by E^S, the set of all weak edges will be denoted by E^W. E^S and E^W are a partition of E, that is $E^S{\cup}E^W{=}E$ and $E^S{\cap}E^W{=}\varnothing$. It happens that all edges of H_T are strong, for example, when only one feasible placement exists, namely $\varphi{=}0$ (that is $\lambda_i = r_i{-}l_i$ for all $t_i{\in}T$).

A strong edge is contained in all interval graphs $G{=}(V,E_G)$ of the family $\mathcal{F}_T$, since the corresponding intervals intersect whatever their (feasible) placement is. Thus $E^S = \bigcap_{G\in\mathcal{F}_T} E_G$. On the contrary, a graph of the family $\mathcal{F}_T$ contains a weak edge of H_T if the corresponding intervals are "suitably" placed within their window. Clearly, given a weak edge (u,v) of H_T, there always exist two distinct graphs in the family, one which does contain the weak edge (u,v), and the other one which does not.

From what above, it follows that the family $\mathcal{F}_T$ has a positive finite cardinality, in fact $1 \leq |\mathcal{F}_T| \leq 2^{|E^W|} < 2^{n^2}$. Let $r_{max} = \max\{r_i, i{=}1,\ldots,n\}$ and $l_{min} = \min\{l_i, i{=}1,\ldots,n\}$. Since every φ_i may assume one out of $r_i{-}l_i{-}\lambda_i{+}1$ integer consecutive non–negative values, we get $O((r_{max}{-}l_{min})^n)$ interval models, which are mapped onto the set $\mathcal{F}_T$ whose cardinality does not exceed 2^{n^2}. Notice that if H_T does not contain weak edges, one has $|\mathcal{F}_T| = 1$, and the unique graph $G{\in}\mathcal{F}_T$ is exacty H_T.

Moreover:

Observation 2.2: Any interval graph $G{=}(V,E_G){\in}\mathcal{F}_T$ is a partial subgraph of $H_T{=}(V,E)$, in the sense that $E^S \subseteq E_G \subseteq E{=}E^S{\cup}E^W$.

On the other hand, not all partial subgraphs $G'=(V,E_{G'})$ of H_T which are interval graphs belong to $\mathcal{F}_T$, even if they verify $E^S \subseteq E_{G'} \subseteq E=E^S \cup E^W$. In Figure 1 an example is discussed.

Definition 2.3: A triple t_v is *overlapping*, and so is the corresponding vertex, if $r_v-l_v < 2\lambda_v$. Triples and vertices are *non–overlapping* in all other cases.

Consider a triple t_v. Notations $<t_v,0>$ and $<t_v,r_v-l_v-\lambda_v>$ represent the interval of the triple t_v in its leftmost and rightmost placement, respectively. The corresponding vertex $v \in V$ is called overlapping if the two intervals $<t_v,0>$ and $<t_v,r_v-l_v-\lambda_v>$ do have non–empty intersection. In this case there exists a set of points of the real line which is contained into both of them, an precisely $(r_v-\lambda_v,l_v+\lambda_v]$. On the contrary, vertex v is non–overlapping when the two intervals $<t_v,0>$ and $<t_v,r_v-l_v-\lambda_v>$ do have empty intersection. In this case there exists a set of points of $(l_v+\lambda_v,r_v-\lambda_v]$ the real line which does not intersect the intervals $<t_v,0>$ and $<t_v,r_v-l_v-\lambda_v>$.

The set V^O of all overlapping vertices and the set V^N of all non–overlapping ones are a partition of the vertex set V of H_T, in the sense that $V^O \cup V^N=V$ and $V^O \cap V^N=\emptyset$.

Lemma 2.4: No strong edge connects two non–overlapping vertices.

Proof: Consider two non–overlapping vertices u,v, and let $c_u=\frac{r_u-l_u}{2}$, and $c_v=\frac{r_v-l_v}{2}$ be the central coordinate of the corresponding windows. Assume, without loss of generality, that $c_u \leq c_v$. We claim that there exist placements φ_u, φ_v such that the intervals $<t_u,\varphi_u>$, $<t_v,\varphi_v>$ do not intersect. In fact, since, by hypothesis, both vertices are non–overlapping, interval u placed in its leftmost position (i.e., $<t_u,0> = (l_u,l_u+\lambda_u]$) lays completely on the left of c_u, and interval v placed in its rightmost position (i.e., $<t_u,r_v-l_v-\lambda_v> = (r_v-l_v r_v]$) lays completely on the right of c_v That is, $l_u+\lambda_u \leq c_u$ and $c_v \leq r_v-\lambda_v$, and the theorem is proved. ❑

Theorem 2.5: Let v be a non–overlapping vertex, and let $\mathrm{Adj}^S(v)$ denote the set of all vertices connected to v by a strong edge. Then all vertices in $\mathrm{Adj}^S(v)$ are overlapping and are pairwise connected by strong edges.

Proof: Consider any two vertices $x,y \in \mathrm{Adj}^S(v)$. Since edges (x,v), (y,v) are strong, both x and y are overlapping, in light of Lemma 2.4. Moreover, whatever φ_x, φ_y, and φ_v are, intervals $<t_x,\varphi_x>$ and $<t_y,\varphi_y>$ intersect interval $<t_v,\varphi_v>$, by definition; that is both inequalities $r_a-\lambda_a \leq l_v+\lambda_v$ and $r_v-\lambda_v \leq l_a+\lambda_a$ must hold for $a=x,y$. Since v is a non–overlapping vertex, $l_v+\lambda_v < r_v-\lambda_v$. The theorem follows by considering that $(l_v+\lambda_v,r_v-\lambda_v] \subseteq (l_x+\varphi_x,l_x+\varphi_x+\lambda_x]$ and $(l_v+\lambda_v,r_v-\lambda_v] \subseteq (l_y+\varphi_y,l_y+\varphi_y+\lambda_y]$. ❑

Notice that $\{v\} \cup \mathrm{Adj}^S(v)$ induce a complete subgraph whose edges are all strong. Thus it is a subgraph of H_T, as well as of any $G \in \mathcal{F}_T$. From the above theorem we also derive that any complete subgraph of H_T, with all strong edges, has at most one non–overlapping vertex.

Consider the subgraph $A=(V,E^S)$ of H_T defined on the whole set of vertices and on the set E^S of all strong edges, only, and the subgraph $H_T(V^O)$ induced in H_T by the set V^O of all non–overappig vertices only. We notice that $H_T(V^O)$ is a subgraph of A, and that $H_T(V^O)$ is an interval graph, while A is not, generally speaking. Let us consider the following example. Be $T = \{t_1=<0,10,8>, t_2=<3,28,5>, t_3=<2,11,3>, t_4=<6,9,2>, t_5=<12,18,3>\}$ (see Fig. 1 and 2).

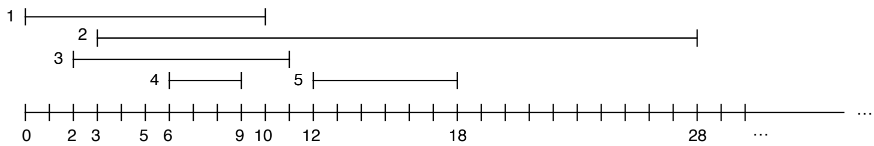

Figure 1 – Example: the windows of *T*.

In Fig.2(a) the graph H_T is drawn, where the black (white) vertices are overlapping (non–overlapping) and the thick (thin) edges are strong (weak)). We note that vertices 1 and 4 are overlapping, and edge (1,4) is the only strong edge. In Fig. 2(b), 2(c), 2(d) some examples of partial subgraphs of H_T defined on the whole set of nodes.

In Fig.2(b) a graph G of the family $\mathcal{F}_T$ is drawn; it is an interval graph and it does contain the only strong edge; a feasible placement φ such that $G(\varphi)$ is isomorphic to G, is the following $\varphi=(0,0,6,1,3)$; this is not the unique φ with such property, any other $\varphi=(0,0,6,1,\varphi_5)$, with $\varphi_5\in[0,3]$, gives rise to the graph in the figure.

Figure 2 – Example (continued).

In Fig. 2(c) a partial subgraph of H_T is drawn, which is not interval, nor it contains the only strong edge; for both reasons it can not belong to the family $\mathcal{F}_T$.

In Fig. 2(d) a partial subgraph of H_T is drawn, which is an interval graph, and it also contains the only strong edge; nevertheless it does not belong to $\mathcal{F}_T$ because no feasible placement exists which gives rise to this interval graph, as we are going to discuss. Denote by $G=(V,E_G)$ the graph in the figure, where $E_G=\{(1,4), (2,3), (2,4), (2,5)\}$ (recall that (1,4) is a strong edge). Since $(2,4)\in E_G$ and $(1,2)\notin E_G$, that is, since interval $<t_2,\varphi_2>$ has to intersect interval $<t_4,\varphi_4>$ and <u>not</u> interval $<t_1,\varphi_1>$, it must be the case that $\varphi_1=0$, $\varphi_2=5$, and $\varphi_4=1$. But now, there is no way of choosing φ_3 so that $<t_3,\varphi_3>$ intersects no interval but $<t_2,5>$, (reflecting the fact that edge (2,3) is the only edge incident on v_3 in G). In particular we note that <t_3,φ_3> intersects interval <t_2,5> only if $\varphi_3 \in \{4,5,6\}$. It is easy to verify that when $\varphi_3 \in \{4,5,6\}$, interval <t_3,φ_3> happens to intersect also <t_4,1>, and that when $\varphi_3 \in \{4,5\}$ it intersects <t_1,0>. This contradicts the fact that (3,4), (1,3)$\notin E_G$, showing that $G \notin \mathcal{F}_T$, and proves that the presence/absence of a subset of weak edges in a graph <u>does</u> depend, generally speaking, on the presence/absence of some other subset of weak edges.

This concept can be generalized even further. Let J be the family of all partial subgraphs of H_T which are defined on the whole set of vertices, which contain all the strong edges and any subset of weak edges (including also the empty one), and which are interval graphs. It is the case that $J \subseteq \mathcal{F}_T$, and we showed an example where $J \subset \mathcal{F}_T$. This means that the characterization of J, although quite restrictive, is not sufficient yet to describe the only interval graphs $G(\varphi)$ corresponding to feasible φ's, and that the problem of recognizing if a graph $G \in J$ also belong to $\mathcal{F}_T$, that is if there exists a feasible φ such G is isomorphic to $G(\varphi)$, is still open. This allows us for concluding that neither the definition of weak edge, neither considering all possible subsets of them are "complex" enough for correctly describing all graphs in $\mathcal{F}_T$, even if we restrict our attention to the only subsets of weak edges which, together with all the strong edges, give rise to interval graphs.

From the above observations it follows that the problem of finding a particular $G \in \mathcal{F}_T$ is equivalent to that of finding a feasible placement which induces G.

Definition 2.6: A triple set T is *orderable* if $(l_i,r_i] \neq (l_j,r_j]$ for $i \neq j$, and no window exists which is properly contained into another one.

In this case, the triples of T, hence the vertices of H_T, can be numbered in such a way that $i<j$ if and only if $l_i \leq l_j$ and $r_i \leq r_j$. In what follows we shall always assume that the triples of an orderable triple set T are numbered according to this criterion, as well as the vertices of the corresponding interval graph H_T.

Notice that the intersection graph H_T of the windows of an orderable triple set T is a proper interval graph, and proper is also the set of windows. In fact, a set of intervals is proper if no interval exists which is properly contained into another one, and an interval graph G is proper if there exists a set of intervals none of which is properly contained into another one, and whose intersection graph is G (a complete characterization of proper interval graphs, also known as *unit interval graphs* can be found in (Golumbic 1980)).

3. The clique number and chromatic number problems

As noticed before, once a feasible placement φ is given, the intersection relation among the intervals in $M(\varphi)$ can be represented by means of a graph $G(\varphi) \in \mathcal{F}_T$, which clearly is an interval graph, and the equality $\omega(G(\varphi))=\chi(G(\varphi))$ holds for it (Golumbic 1980). This implies that the result of the optimization of $\omega(T)$ immediately reflects onto the same optimization of $\chi(T)$.

3.1 *Minimization of* $\omega(T)$ *and* $\chi(T)$

The present Section is devoted to the problem Min $\omega(G)$ of minimizing $\omega(G)$ over all $G \in \mathcal{F}_T$.

3.1.1 Computational complexity

Consider problem Min $\omega(T)$ in decision form.

Theorem 3.1: Problem Min $\omega(T)$ in decision form, where the given triple set T is such that $l_i = 0$ and $r_i = \rho$ for all $i = 1,\ldots,n$, is NP-complete in the strong sense.

Proof: The problem is easily seen to be in NP. The proof is by reduction from 3-PARTITION, which is NP–complete in the strong sense (Garey and Johnson 1979). Given $3m$+1 positive integers $b_1, \ldots, b_{3m}$, B with $\frac{B}{4} < b_j < \frac{B}{2}$ for any $j = 1, \ldots, 3m$ and $\sum_{j=1,\ldots,3m} b_j = mB$, find a partition of the b_j's into m subsets $\beta_1,\ldots,\beta_m$, such that $\sum_{b_j \in \beta_i} b_j = B$ and $|\beta_i| = 3$ for any $i = 1,\ldots,m$. Now, given any instance of 3-PARTITION we construct in polynomial time a corresponding instance of Min $\omega(T)$ in decision form, as follows: we set $h = m$, and with each integer b_j we associate a triple $t_j = \langle l_j, r_j, \lambda_j \rangle$, where $l_j = 0$, $r_j = B$ and $\lambda_j = b_j$, for any $j = 1,\ldots,3m$.

Assume the resulting instance of Min $\omega(T)$ in decision form is a YES-instance, and let φ be a feasible placement such that $\omega(G\varphi)) \leq h$, it is to say, each unit range $(j,j+1]$ for j=0,…,B–1 is contained into no more than h intervals. Since the total length of the intervals amounts to hB, and the horizontal dimension of the problem is bounded by B, we can conclude that the above relations hold with the equality sign, namely, each unit range $(j,j+1]$ for j=0,…,B–1 is contained into exactly h intervals (that is, all complete subgraph of $G(\varphi)$ are maximal w.r.t. node inclusion and have size (exactly) h). The corresponding YES–solution for 3–PARTITION is obtained this way. Let L be the set of the h intervals whose left endpoint has value zero, let R be the set of the h intervals whose right endpoint has value B, and let C be the set of the remaining h ones. Since any unit range $(j,j+1]$ for j=0,…,B–1 is contained into exactly h intervals, for any $i \in C$ there exist exactly two intervals $\langle t_{i'}, \varphi_{i'} \rangle$ with $i' \in L$ and $\langle t_{i''}, \varphi_{i''} \rangle$ with $i'' \in R$ such that the left endpoint of $\langle t_i, \varphi_i \rangle$ coincides with the right endpoint of $\langle t_{i'}, \varphi_{i'} \rangle$, and the right endpoint of $\langle t_i, \varphi_i \rangle$ coincides with the left endpoint of $\langle t_{i''}, \varphi_{i''} \rangle$, that is $\varphi_i = \lambda_{i'}$ and $\varphi_i + \lambda_i = \varphi_{i''}$ (in fact $\varphi_i = 0$ for any $i \in L$ and $\varphi_i = B - \lambda_i$ for any $i \in R$). Since, clearly, $\lambda_{i'} + \lambda_i + \lambda_{i''} = B$ for $i=1,\ldots,h$, $\beta_i = \{\beta_{i'}, \beta_i, \beta_{i''}\}$ for $i=1,\ldots,h$ is the desired YES–solution for the given YES–instance of 3–PARTITION ($h=m$ by construction).

Assume, on the contrary, that the resulting instance of Min $\omega(T)$ in decision form is a NO-instance, that is, no feasible placement φ exists such that $\omega(G(\varphi)) \leq h$. For any feasible placement φ we have $\omega(G(\varphi)) > h$, that is some unit range happens to be contained into more than h intervals. Since the total length of the intervals amounts to hB, we can conclude that some other unit range happen to be contained into less than h intervals. In this case we claim that no way exists of partitioning the set of intervals into h subsets of three mutually non overlapping elements each, because not all the just required constraints can be satisfied at once. In fact, consider, for example, any integer $j \in [0, B-1]$ such that the unit range $(j, j+1]$ is contained into $h' > h$ intervals. Clearly, in order to have subsets of non overlapping intervals, each one of the h' intervals has to be assigned a different subset, therefore we will have more than h subsets, contradicting one constraint. On the other hand, from the instance data, we immediately derive that neither all such subsets will have 3 elements, nor the sum of the length of the intervals assigned to each of them amounts to B, proving the theorem. ❑

The following corollary is a trivial consequence of the previous theorem.

Corollary 3.2: Given an arbitrary triple set T, the optimization version of problems Min $\omega(T)$ and Min $\chi(T)$ are NP-hard in a strong sense.

3.1.2 Polynomially solvable cases

Even though problem Min $\omega(T)$ in optimization form has just been proved to be NP-hard on arbitrary triple sets, there are instances whose particular structure allows for finding an optimum solution in polynomial time.

Let us start by considering the decision version of problem Min $\omega(T)$. Algorithm mM-C (min Max Clique) described below outputs a feasible solution, if any, to instances defined on orderable triple sets T (triples are numbered by non–decreasing left endpoints) such that the sequence of values $r_i - \lambda_i$, $i=1,\ldots,n$, is non–decreasing (that is $r_i - \lambda_i \geq r_{i-1} - \lambda_{i-1}$, for $i=2,\ldots,n$). Let W be a set of intervals, open on the left; throughout the algorithm, $c_h(W)$ will denote the largest coordinate contained into exactly h intervals of W ($c_h(W) = 0$ if $W = ø$ or no coordinate exists which belongs to exactly h intervals of W) (see Fig. 3).

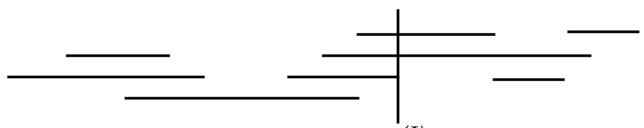

Fig. 3 — $c_3(W)$, where W is the depicted set of intervals open on the left

```
Algorithm mM-C;
Input:      an orderable triple set T = {t1,t2,…,tn} such that ri–λi ≥ ri–1–λi–1, for i=2,…,n,
            and a non–negative integer h;
Output:     a feasible placement φ such that ω(G(φ)) ≤ h;

Begin
For i = 1,…,n Do
        φi := undefined;
Let W := ø;
For i = 1,…,n Do
        Begin
        If ch(W) > ri – λi
```

```
                Then Return(NO) and Stop
                Else   Begin
                       φi = max {0,  ch(W) − li};
                       W := W ∪ {(ti,φi)};
                       End;
                End;
        Return(YES);
        End.
```

Theorem 3.3: Algorithm mM-C correctly solves problem Min $\omega(T)$ in decision form on orderable triple sets $T = \{t_1,t_2,\ldots,t_n\}$ such that $r_i-\lambda_i \geq r_{i-1}-\lambda_{i-1}$, for i=2,…,n.

Proof: The correctness of the algorithm is obtained proving the following two facts: *i*) the clique number of the intersection graph of the set of intervals placed according to Algorithm mM–C does not exceed h, and *ii*) the algorithm returns NO if and only if the given SIG is a NO–instance.

The first claim is immediate: in fact the algorithm stops as soon as the distance between the current $c_h(\cdot)$ and the right endpoint of the window under consideration is smaller than the corresponding interval length.

The "if"–part of the second claim follows from the fact that no feasible placement exists for the given instance.

Let us prove the "only if" part of the second claim. Assume, by contradiction, that the given instance of problem Min $\omega(T)$ in decision form is a YES–instance. Let s+1 be the last interval considered by the algorithm when it happens to return NO, that is $i = s+1$, $W \equiv \{<t_1,\varphi_1>,\ldots,<t_s,\varphi_s>\}$, and, of course, $c_h(W) > r_{s+1} - \lambda_{s+1}$. Since the given one is assumed to be a YES-instance, there exists a feasible placement $\tilde{\varphi}$ such that interval s+1 happens to belong to a clique of cardinality not larger than h. We shall prove that no such placement $\tilde{\varphi}$ exists, contradicting the hypothesis.

Since the given triple set T is orderable and $r_i-\lambda_i$, for i=1,…n, is a sequence of non–decreasing values, we are allowed to limit ourselves to modifying the current placement φ of intervals 1 through s+1 in order to determine the first s+1 components of $\tilde{\varphi}$. Let J be the set of the indices of the intervals which, placed according to φ, contain coordinate $c_h(W)$, that is $J = \{a\leq s,\ l_a+\varphi_a < c_h(W)\leq l_a+\varphi_a+\lambda_a\}$. In order to place interval s+1 so that the clique number of the resulting interval graph does not exceed h, we shall try to shift some interval $j\in J$. Three cases arise:

a) no interval $j\in J$ exists which can be shifted (neither rightward nor leftwards) so as to lay fully within its window and completely to the right of $c_h(W)$, or completely to the left of the left endpoint of interval s+1 in its rightmost placement; that is, no feasible placement φ'_j exists so that $l_j+\varphi'_j \geq c_h(W)$ or $l_j+\varphi'_j+\lambda_j \leq r_{s+1}-\lambda_{s+1}$;

b) there exists an interval $j\in J$ which can be shifted rightwards so as to lay fully within its window and completely to the right of $c_h(W)$; that is, there exists a feasible placement $\varphi'_j > \varphi_j$ such that $l_j + \varphi'_j \geq c_h(W)$;

c) there exists an interval $j \in J$ which can be shifted leftwards so as to lay fully within its window and completely to the left of the left endpoint of interval s+1 in its rightmost placement; that is, there exists a feasible placement $0 \le \varphi'_j < \varphi_j$ such that $l_j + \varphi'_j + \lambda_j \le r_{s+1} - \lambda_{s+1}$.

Assume case *a*) applies. Then the given one is a NO–instance, and in fact no feasible placement exists such that the number of intervals crossing coordinate $c_h(W)$ is smaller than h+1.

Assume case *b*) applies. Since $c_h(W) \le l_j+\varphi'_j \le r_j-\lambda_j$, we have that $c_h(W)-l_j \le \varphi'_j \le r_j-l_j-\lambda_j$. On the other hand, since the given triple set T verifies $r_i-\lambda_i \ge r_{i-1}-\lambda_{i-1}$, for i=2,…,n, then $r_j-\lambda_j \ge c_h(W)$ implies that $r_{s+1}-\lambda_{s+1} \ge c_h(W)$. This contradicts the stopping condition of the algorithm, and shows that $\tilde{\varphi}_{s+1} = c_h(W) - l_{s+1}$ is a feasible placement for interval s+1 which does not increase over h the number of mutually intersecting intervals.

Assume case *c*) applies. Because of the algorithm behavior, if we shift leftwards an interval, we have to shift rightwards another one, in order not to increase over h the number of mutually intersecting intervals on the left of $c_h(W)$. This means that we have to find an interval with index $i<j$ which has to be moved rightwards in order to allow the leftwards shifting of j. Let $W' \equiv \{<t_1,\varphi_1>, \ldots, <t_{j-1}\varphi_{j-1}>\}$ and let J' be the set of the indices of all the intervals p which intersect coordinate $c_h(W')$, with $p<j$. The same kind of reasoning seen above on the possible shifting of an interval can be iteratively applied, until case *a*) applies (which is always the case, for example when $c_h(W') > 0$ for the first time during the execution of the algorithm), which ends the proof. Notice that $c_h(W')$ is always greater than 0, otherwise case a) would apply. ❑

The computational complexity of algorithm mM-C is O(n) if we assume that the windows are already sorted.

The optimization version of problem Min $\omega(T)$ can be solved by applying Algorithm mM-C to a sequence of problems with different values of k. A good way of operating consists of applying a dichotomic search on the values of k, as lower and upper bounds are known for it. Namely, one can apply a dichotomic search for k in the range $[1,\omega(H_T)]$ since $1 \le \min \omega(T) \le \max \omega(T) \le \omega(H_T) \le n$. The number of steps of the dichotomic search amounts to O(log $\omega(H_T)$) which is bounded from above by O(log n). This finally gives an O(n log n) time algorithm to solve the optimization version of problem Min $\omega(T)$ on orderable triple sets which verify $r_i-\lambda_i \ge r_{i-1}-\lambda_{i-1}$, for i=2,…,n.

3.1.3 Lower and upper bounds

Let $A=(V,E^S)$ be the subgraph of H_T defined on the whole set of vertices and on the set E^S of all strong edges, only.

Theorem 3.4: $\omega(A) \le \omega(T)_{min} \le \omega(H_T)$.

Proof: The first inequality follows immediately considering that, by definition, A is a subgraph of any graph $G\in\mathcal{F}_T$; the second one, by considering that $\omega(G) \leq \omega(H_T)$ for any $G\in\mathcal{F}_T$, as every G is a subgraph of H_T ❑

We now describe how to compute $\omega(A)$ (recall that A is not an interval graph, generally speaking). Since, by Theorem 2.5, any complete subgraph with all strong edges contains at most one non–overlapping vertex, we can reason as follows. We compute $|\{v\}\cup \mathrm{Adj}^S(v)|$ for each non–overlapping vertex v. Next we consider the interval subgraph A' induced in A by the vertex set $V^O \setminus (\bigcup_{v\in V^N} \mathrm{Adj}^S(v))$, and compute $\omega(A')$ (Gupta et al. 1982). It results $\omega(A) = \max\,\{\omega(A'), \max_{v\in V^N}\{|\{v\}\cup \mathrm{Adj}^S(v)|\}\}$.

3.2 *Maximization of $\omega(T)$ and $\chi(T)$*

In this section we shall deal with the problem of maximizing $\omega(G)$ over the set of all graphs $G\in\mathcal{F}_T$. Unlike the problem Min $\omega(T)$, problem Max $\omega(T)$ on arbitrary triple sets can be solved quite easily, both in decision and in optimization form.

Clearly, $\omega(G) \leq \omega(H_T)$ for any $G\in\mathcal{F}_T$, as every G is a subgraph of H_T. We shall show in a while that there always exists a graph G of the family whose clique number equals the clique number of H_T, that is $\max\,\{\omega(G), G\in\mathcal{F}_T\} = \omega(H_T)$. The placement vector φ which gives rise to a graph with such property is obtained this way: be x any coordinate belonging to exactly $\omega(H_T)$ windows (such a coordinate does always exist); let $C = \{i: l_i<x\leq r_i\}$ be the set of indices of the windows containing x; and consider any feasible placement vector φ that verifies $x-l_i-\lambda_i \leq \varphi_i < x-l_i$ for all $i\in C$. By construction, all intervals $<t_i,\varphi_i>$, with $i\in C$, intersect coordinate x, that is, $\omega(G(\varphi)) = \omega(H_T)$, as desired.

As a consequence, the complexity of determining the placement vector φ which maximizes $\omega(T)$ is dominated by the complexity of determining a complete subgraph of H_T with maximum size, which requires $O(n \log n)$ (Gupta et al. 1982).

4. The stability number and clique cover number problems

Let T be a triple set. As any $G\in\mathcal{F}_T$ is an interval graph, and equation $\alpha(G)=k(G)$ holds for it, the result of the optimization of $\alpha(T)$ immediately applies to the same optimization of $k(T)$.

4.1 *Maximization of $\alpha(T)$ and $k(T)$*

The present Section is devoted to the problem Max $\alpha(T)$ of maximizing $\alpha(G)$ over all $G\in\mathcal{F}_T$.

4.1.1 Computational complexity

Consider problem Max $\alpha(T)$ in optimization form.

Theorem 4.1: The optimization version of problem Max $\alpha(T)$ is NP-hard.

Proof: Consider the one-machine n jobs scheduling problem with ready and due times. The problem of minimizing the number of tardy jobs is NP-hard when the ready times are non-negative (Lenstra et al. 1977). This problem can be trivially reduced to Max $\alpha(T)$ on the triple set T, which is the collection of all triples $t_i = <\rho_i,\delta_i,p_i>$, $i = 1,\ldots,n$ where ρ_i, δ_i, p_i are the release date, the due date, and the processing time of job i, respectively. The size of a maximum independent set over all graphs $G\in\mathcal{F}_T$ is equal to the number of jobs processed on time. ❑

The following corollary is an immediate consequence of the previous theorem.

Corollary 4.2: The optimization version of problem Max $k(T)$ is NP-hard.

4.1.2 Polynomially solvable cases

Despite the complexity of problem Max $\alpha(T)$ in the general case, there are sets of triples whose particular structure gives rise to instances of the problem solvable in polynomial time.

The maximization of $\alpha(T)$ on orderable triple sets can be conducted in $O(n^2)$ time using the algorithm by Kise et al. (Kise et al. 1978), which is based on a dynamic programming approach.

However, if the given (orderable) triple set is such that such that $\lambda_i \geq \lambda_{i-1}$, for i=2,…,n, an $O(n)$ time algorithm is proposed for its solution. We shall assume that triples are numbered by non–decreasing left endpoints.

```
Algorithm MM-IS
Input:    an orderable triple set T = {t1,t2,...,tn} such that λi ≥ λi−1, for i=2,...,n;
Output:   a feasible placement φ such that α(G(φ)) is maximum and a maximum sized
           independent set Y;

Begin
For i = 1,...,n Do
    φi := 0;
Let Y := ø;
p := l1;
For i = 1,...,n do
    Begin
    If p ≤ ri − λi
    Then  Begin
          φi := max {0,p − li};
          Y := Y ∪ {<ti,φi>};
          p := li + φi + λi
          End
    End
End.
```

Notice that Y is a maximum sized independent set for all graphs $G(\tilde{\varphi})$ where $\tilde{\varphi}_i$=φ_i for all i such that $<t_i,\varphi_i>\in Y$, and $\tilde{\varphi}_i$ feasible in all other cases.

Theorem 4.3: Algorithm MM-IS for orderable triple sets which verify $\lambda_i \geq \lambda_{i-1}$, for i=2,…,n, is correct.

Proof: The algorithm proceeds by defining a sequence of (nested) subproblems, each defined on the first i triples, for i=1,…,n. Iteratively, it finds a maximum independent set Y for the current subproblem. It can be observed that among all possible independent sets of maximum size for the current subproblem $T' \equiv \{t_j: j = 1,\ldots,i\}$, one is proposed whose rightmost right endpoint is minimum. We now justify this claim. The proof is by induction. For $i = 1$ we have $Y = \{(t_1,0)\}$ and the above observation holds. Assume that the observation is true for i and consider i+1. Either interval i+1 can be added to Y and the observation holds true, or interval i+1 does not admit a feasible placement which allows its insertion into the independent set Y. In this second case we could either discard interval i+1 or discard another interval $h < i$+1 and possibly insert i+1 into Y with a suitable feasible placement; in this latter subcase, in particular, one has to find a new placement to all intervals h+1,…,i and then possibly place i+1 in a feasible position, so as to insert it into Y. But, since the given triple set is orderable and $\lambda_j \geq \lambda_{j-1}$, for j=2,…,n, one has $\lambda_{i+1} \geq \lambda_i$, this second case would produce an Y of the same size to which corresponds a value of p which can not be smaller than the one corresponding to the previous Y. ❑

If the intervals are already sorted, the complexity of MM-IS is, clearly, O(n).

4.1.3 Lower and upper bounds

Let $A=(V,E^S)$ be the subgraph of H_T defined on the whole set of vertices and on the set E^S of all strong edges, only.

Theorem 4.4: $\alpha(H_T) \leq \alpha(T)_{max} \leq \alpha(A)$.

Proof: The inequalities follow immediately recalling that H_T and A are supergraph and subgraph, respectively, of any graph $G \in \mathcal{F}_T$. ❑

We now describe how to construct and independent set of maximum size for graph A (recall that A is not an interval graph, generally speaking). Let X be a maximum independent set of the interval subgraph A' induced in A by the vertex set $V^O \backslash (\bigcup_{v \in V^N} \mathrm{Adj}^S(v))$ (Gupta et al. 1982). We claim that $V^N \cup X$ is an independent set of maximum size for graph A. By Theorem 2.5, only one vertex out of $\{v\} \cup \mathrm{Adj}^S(v)$ belongs to an independent set of graph A, for all $v \in V^N$, and, by Lemma 2.4, V^N is an independent set of graph A. By construction, no vertex in X is adjacent to any vertex in V^N. On the other hand, no way of replacing any subset of vertices of V^N with any larger sized subset of vertices of X exists. Thus $\alpha(A) = |V^N \cup X|$.

4.2 *Minimization of* $\alpha(T)$ *and* $k(T)$

In this section we discuss problem Min $\alpha(T)$, that is the problem of finding a feasible placement φ which defines an interval graph $G(\varphi)$ whose maximum independent set has minimum size among all the graph of the family $\mathcal{F}_T$.

The solution to this problem is easily found thanks to the well-known equality holding among the stability number $\alpha(\cdot)$ and the clique cover number $k(\cdot)$ of a perfect graph. Consider the intersection graph H_T of the set of windows; be Γ a minimum covering by cliques of the node set V of H_T (Gupta et al. 1982); for each clique $C\in\Gamma$, let x be a coordinate contained into every window whose corresponding node is in C; place any interval $i\in C$ so as to contain coordinate x, that is, choose a feasible φ_i so that $l_i+\varphi_i < x \leq l_i+\varphi_i+\lambda_i$. If there exists windows whose corresponding node belongs to more than one clique, break ties arbitrarily. It is easy to see that such an algorithm, whose running time is linear in the number n of triples, finds a placement vector φ whose corresponding graph $G(\varphi)$ has a minimum sized cover by cliques of minimum cardinality (thus also a maximum sized independent set of minimum cardinality) among all the graphs in the family $\mathcal{F}_T$.

5. The dominating set problem

A subset of nodes $D(G)\subseteq V$, of cardinality $d(G)$, is a *dominating set* for an arbitrary graph $G = (V,E)$, iff for any $u\in V\setminus D(G)$ there exists a $v\in D(G)$ such that edge $(u,v)\in E$. The optimization goal is usually finding a minimum sized dominating set, since the problem of finding one with maximum size is trivial (it is V). The dominating set problem on interval graphs and on chordal graphs (interval graphs are chordal) is dealt with in (Gupta et al. 1982, Booth and Johnson 1982, Farber 1985, Ramalingan Rangan 1988).

Both problems Min $d(T)$ and Max $d(T)$ can be defined, consisting of identifying an interval graph $G\in\mathcal{F}_T$ the size of a minimum sized dominating set of which is minimum (respectively, maximum) among all the graphs in the family. In the sequel, we shall deal with the min–min type problem Min $d(T)$, and $D(T)_{min}$ and $d(T)_{min}=|D(T)_{min}|$ will denote a minimum dominating set for the given triple set T, and its cardinality, respectively.

5.1 *Minimization of* $d(T)$

5.1.1 Computational complexity

In this section we shall prove the strong NP-completeness of the Min $d(T)$ problem. For this reason, throughout the present section we shall refer to the decisional version of the problem.

Theorem 5.1: Problem Min $d(T)$ in decisional form is NP–complete in the strong sense.

Proof: The problem is immediately seen to be in NP. The reduction is from 3–PARTITION, which is NP–complete in the strong sense (Garey and Johnson 1979): given a nonnegative integer B, and a finite set $A = \{a_1, a_2, \ldots, a_{3m}\}$ of $3m$ integers such that $\frac{B}{4} < a_i < \frac{B}{2}$, for $i = 1, \ldots, 3m$, and such that $\sum_{a_i \in A} a_i = mB$, find a partition of A into (exactly) m disjoint sets $A_1, A_2, \ldots, A_m$ such that $\sum_{a_i \in A_j} a_i = B$ for all $j = 1, \ldots, m$ (in order to avoid some trivial cases assume $B \geq 6$). Notice that by the preceding hypothesis it follows that every set has exactly 3 elements. From a given instance of 3–PARTITION construct the following instance of Min $d(T)$ in decisional form. h is set to $3m$ and the triple set T is the union of two sets T_1 and T_2, where $T_1 = \{t_i = \langle 0, m(B+7)-1, a_i \rangle$: for $i = 1, \ldots, 3m\}$ and $T_2 = \{t_j = \langle j, j+1, 1 \rangle$: for $j = 0, \ldots, m(B+7)-2$, with $j \neq k(B+7)-1$ for $k = 1, \ldots, m-1\}$. In what follows, the windows and intervals of triples in T_1 will be called *large*, and the windows and intervals of triples in T_2 will be called *small*. In other words, T_2 consists of m sequences of $B+6$ unit windows with corresponding unit length intervals, each sequence being separated by the following one by a unit space (*jump*), while T_1 consists of $3m$ large windows within each of which an interval of length a_i is to be placed. Notice that each large window properly contains all the small windows, and the only feasible placement for all small intervals s is $\varphi_s = 0$. We claim that there is a dominating set for the given set of triples T with cardinality not larger than $h = 3m$, if and only if a partition with the required properties exists for 3–PARTITION. The IF part follows immediately by observing that a dominating set with cardinality $h = 3m$ is easily obtained from a (feasible) solution $A_j = \{a_{j_1}, a_{j_2}, a_{j_3}\}$, with $j = 1, \ldots, m$, for 3–PARTITION, by setting $\varphi_s = 0$ for all small intervals, and by setting $\varphi_{j_1} = (j-1)(B+7)+1$, $\varphi_{j_2} = (j-1)(B+7)+a_{j_1}+3$, $\varphi_{j_3} = (j-1)(B+7)+a_{j_1}+a_{j_2}+5$ for all $j = 1, \ldots, m$. Let us prove the ONLY IF part. We shall say that the i^{th} large interval is *nicely placed* if φ_i is such that no jump is contained in $[\varphi_i, \varphi_i+a_i]$ and there are two small windows w_s, w_t such that $r_s = \varphi_i$, and $l_t = \varphi_i+a_i$. It is easy to see that the i^{th} large interval may dominate up to a_i+2 small windows, such maximum value being achieved if and only if it is nicely placed. Since the assumed hypothesis $B \geq 6$ implies $a_i > 1$ for all $i = 1, \ldots, 3m$, it is immediate to see that any dominating set of cardinality not larger than $3m$ is made of all large intervals, only, and all of them are nicely placed. More precisely, notice that for the same reason no dominating set for T exists with cardinality strictly less than $3m$. The facts that all large intervals are nicely placed and that $\frac{B}{4} < a_i < \frac{B}{2}$, for $i = 1, \ldots, 3m$, implies that the small intervals of a same sequence are dominated by exactly three large intervals. The corresponding 3–PARTITION is obtained by inserting into the same subset A_j the three intervals which dominate the small windows of the j^{th} sequence. Since the reduction from 3–PARTITION to Min $d(T)$ is pseudopolynomial, and 3–PARTITION is strongly NP-complete, the result is proved. ❑

5.1.2 Polynomially solvable cases

This section is devoted to study special cases solvable at optimality. We propose a greedy algorithm, called Greedy and characterize the special classes of triples on which it finds the optimal solution. For the sake of simplicity we shall describe the algorithm on the interval model of the windows, each time suitably positioning the corresponding interval (in the algorithm $T \backslash D$ is an improper writing for $T \backslash \{t_j: <t_j, \varphi_j> \in D\}$).

```
Algorithm Greedy:
Input:   a set of triples T = {t1,t2,...,tn};
Output:      a feasible placement φ and a subset D;

Begin
Initially all the windows are unmarked and the set D is empty.
Repeat
   Consider the leftmost right endpoint π of an unmarked window of T\D ;
   Let X = {i: li≤π≤ri, i ∉ D};
   Set φi = min {ri–li–λi, π–li} for all i∈X;
   Let j∈X be such that lj+φj+λj = max {li+φi+λi, for i∈X};
   Insert j into D;
      Mark all windows wi verifying li ≤ lj+φj+λj ;
Until all windows are marked
End.
```

The algorithm produces a feasible placement φ and a subset D of indices. The subset D is a dominating set for the interval graph $G(\varphi)$. As for the algorithm behaviour we notice what follows: (*i*) by setting $\varphi_i = \min\{r_i - l_i - \lambda_i, \pi - l_i\}$ for all $i \in X$ we are actually placing each interval in the rightmost position within its window, so as to make it cross π; (*ii*) the φ_j are usually set more than once until either j is inserted into D or it does not belong to the current X anymore; (*iii*) at each iteration we insert into the current set D the index of an interval with rightmost right endpoint.

The computational complexity of Algorithm Greedy amounts to $O(n^2)$. In fact, in the worst case the algorithm considers $O(n)$ different subsets X, whose cardinality is bounded by n, and the choice of j takes globally $O(n^2)$ throughout the whole execution of the algorithm.

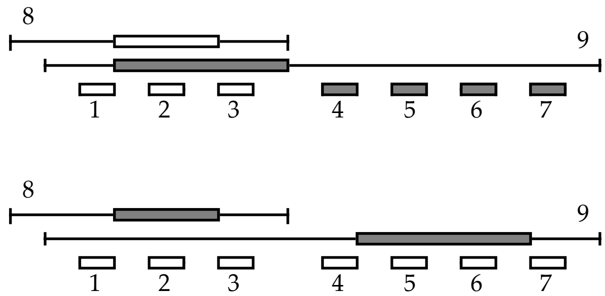

Fig. 4 – Solution output by Algorithm Greedy (above) and an optimum solution (below).

Algorithm Greedy cannot guarantee to determine the optimal solution on arbitrary triple sets since the problem is NP-Hard in the strong sense (Theorem 5.1). An example with

$T = \{t_1$=<3,4,1>, t_2=<5,6,1>, t_3=<7,8,1>, t_4=<10,11,1>, t_5=<12,13,1>, t_6=<14,15,1>, t_7=<16,17,1>, t_8=<1,9,3>, t_9=<2,18,5>$\}$ is shown in Fig. 4: above the solution output by Algorithm Greedy, below an optimum one (the thin rectangles represent the placed intervals, and in particular the grey ones are those in the dominating set).

However there are special classes of triples on which Algorithm Greedy determines an optimum solution, as illustrated in the sequel.

Definition 5.2: A set of triples is *good* if throughout the execution of Algorithm Greedy the current π is greater than the right endpoint r_j of the window corresponding to the last index inserted into D.

Theorem 5.3: Let T be a good set of triples. Then, the placement vector φ output by Algorithm Greedy is an optimal solution to Min $d(T)$.

Proof: By contradiction. Assume that there exists a placement φ^* such that a minimum dominating set D^* on $G(\varphi^*)$ has cardinality smaller than $|D|$. Let K, K^* denote the set of intervals corresponding to indices in D, D^* placed according to φ, φ^*, respectively. The proof consists in showing that taken any coordinate τ the number of intervals of K whose right endpoint lays on the left of τ is never smaller than the number of intervals of K^* whose right endpoint lays on the left of the same point τ, the contradiction being found in the fact that D^* is not a dominating set for $G(\varphi^*)$. For the sake of simplicity we shall denote with $l(s)$, $r(s)$ the left and right endpoint of an interval $s \in K \cup K^*$, namely $l(s) = l_s+\varphi_s$ and $r(s) = l_s+\varphi_s+\lambda_s$.
Sort the intervals in K(K^*, respectively) by non decreasing right endpoint, resulting in the sequence $x_1, \ldots, x_{|K|}$ ($y_1, \ldots, y_{|K^*|}$). Let also w_a be the window with leftmost right endpoint r_a in the given triple set. Consider x_1 and y_1, it must be the case that $l(x_1)$, $l(y_1) \le r_a$, otherwise node a would not be dominated contradicting the hypothesis. Because of the algorithm behaviour, it is also the case that $r(y_1) \le r(x_1)$.
Now consider the next pair x_2, y_2. We shall prove that $r(y_2) \le r(x_2)$. Infact: after having fixed the position of x_1, the algorithm moves to the leftmost right endpoint r_b of a not-yet dominated window w_b (thus $r(x_1) < l_b$), and sets the placement φ_2 of x_2, resulting in $l(x_2) \le r_b \le r(x_2)$. Notice that the algorithm chooses x_2 in the set $C(r_b) = \{i$: such that $l_i \le r_b \le r_i\}$, which x_1 does not belong to, as by hypothesis $r_b > r_{x_1}$. As $r(y_1) \le r(x_1)$, clearly, window w_b is not dominated by y_1. However there must exist another interval in K^* which dominates window w_b. Indeed such interval must be y_2 and one has $l(y_2) \le r_b$. If $r(y_2) \le r_b$, it is also $r(y_2) \le r(x_2)$. If $r(y_2) > r_b$ then $y_2 \in C(r_b)$, and the result follows from the algorithm behaviour.
The reasoning can be repeated, always comparing the pair of intervals x_i, y_i, for $i = 3, \ldots, |K^*|$, concluding that $r(y_{|K^*|}) \le r(x_{|K^*|})$. Consider $x_{|K^*|+1}$. The placement of this interval

is set in order to dominate a not-yet dominated window w_C, verifying $r(x_{|K^*|}) < l_c$. This contradicts the hypothesis that K^* is a dominating set, and the claimed thesis follows. ❑

This theorem, clearly, does not allow to know if Algorithm Greedy will output an optimal solution before running it. But, of course, it is a sufficient condition to prove the optimality of the output solution. There are special set of triples which can be proved *a–priori* to be good: on these set of triples we know that Algorithm Greedy will output an optimal solution. This happens, for example, for the triple sets described in the following theorems.

Theorem 5.4: An orderable triple set is good.

Proof: Let i be the last index inserted by the algorithm into the current D, and let j be the index of window with leftmost right endpoint among the not yet marked (i.e. dominated) ones. Then $l_j > l_i+\varphi_i+\lambda_i$. Since, clearly, $l_i+\varphi_i+\lambda_i \geq l_i$, and the set of triples is orderable, we may conclude that $r_j > r_i$, and the claimed thesis follows. ❑

Theorem 5.5: Let $T = \{t_i=<l_i,r_i,\lambda_i>, i = 1,\ldots,n\}$ be a triple set verifying $\lambda_i \geq r_i–l_i–1$, for i=1,…,n. Then T is good.

Proof: Let j denote the last element inserted into D, and i be the index of an unmarked window with leftmost right endpoint r_i (unmarked w.r.t. the current φ and D). One has: $r_i>l_i$, by definition; $l_i \geq l_j+\varphi_j+\lambda_j+1$, as i is unmarked; and $l_j+\varphi_j+\lambda_j+1 \geq r_j$, as $\lambda_j \geq r_j–l_j–1$, by hypothesis. Since $\pi=r_i$, the thesis follows. ❑

Note that in a triple set with the property required in the Theorem every interval can assume either one of at most 2 positions, that is $\varphi_i \in \{0,1\}$ for all $i = 1,\ldots,n$. In this case "most" edges of H_T are strong.

Theorem 5.6: Let T be a triple set such that H_T has only strong edges. Then T is good.

Proof: Let j denote the last element inserted into D, and i be the index of an unmarked window with leftmost right endpoint r_i (unmarked w.r.t. the current φ and D). One has: $r_i > l_i$, by definition; $l_i \geq l_j+\varphi_j+\lambda_j+1$, as i is unmarked; $l_i > r_j$, as all edges are strong, by hypothesis. Since $\pi = r_i$, the thesis follows ❑

The graph H_T contains only strong edges if, for example, the given triple set T verifies $\lambda_i=r_i–l_i$, for i=1,…,n (that is, intervals are actually not shiftable at all), or if $\lambda_i \geq r_i–l_i–1$, for i=1,…,n and every pair i,j of mutually intersecting windows satisfies $\min\{r_i,r_j\} – \max\{l_i,l_j\} \geq 2$.

However, if H_T has only strong edges, in the family $\mathcal{F}_T$ there is only one graph G, which is isomorphic to H_T. Thus the O(n log$n$) algorithm for computing a minimum cardinality dominating set on interval graphs can be used on $G=H_T$ (Gupta et al. 1982, Farber 1982). The intervals corresponding to the windows in the computed dominating set are a minimum sized dominating set for the unique graph $G \in \mathcal{F}_T$, and also an optimal solution to problem Min $d(T)$.

More sophisticated algorithms for solving problem Min $d(T)$ in the general case are proposed in (Bonfiglio et al., 1987), where their behaviour is experimentally compared with that of Algorithm Greedy.

5.1.3 Lower and upper bounds

Motivated by the complexity result we here state lower and upper bounds to the cardinality $d(T)_{min}$ of a minimum dominating set for the given set of triples T. As far as lower bounds are concerned, we state the following result

Lemma 5.7: Let T be a set of triples. Then $|D_H| \leq |D_G|$, for any $G \in \mathcal{F}_T$.

Proof: Let $\mathrm{Adj}_F(K)$ denote the set of nodes adjacent to at least one node of a subset $K \subseteq V$ in a graph $F = (V, E_F)$. The claimed thesis follows immediately: in fact $\mathrm{Adj}_G(K) \subseteq \mathrm{Adj}_H(K)$, as $E_G \subseteq E_H$. ❏

From this Lemma, it immediately follows that

Theorem 5.8: Let T be a set of triples. Then $|D_H| \leq d(T)_{min}$.

An upper bound to $d(T)_{min}$ is now proposed.

Lemma 5.9: Let T be a set of triples. Then $|D_G| \leq \alpha(H)$, for any $G \in \mathcal{F}_T$.

Proof: Let $K = \{K_1, K_2, \ldots, K_{|K|}\}$ be a minimum cardinality covering by cliques of the node set V of interval graph H (i.e., $|K| = k(H)$, where $k(H)$ is the clique cover number of H). W.l.o.g. assume that a consecutive clique arrangement $<K_1, K_2, \ldots, K_{|K|}>$ is given, and let $\pi(K_i)$ be the rightmost coordinate such that $\{w_t : l_t \leq \pi(K_i) \leq r_t\} = K_i$. Take any two consecutive cliques K_i, K_{i+1}. By the maximality of each clique it follows that there exists (at least) a node v_{j_i} belonging to K_i but not belonging to K_{i+1}. Consider the j_ith interval and place it in such a way that it contains coordinate $\pi(K_i)$, that is, choose φ_{j_i} such that $l_{j_i}+\varphi_{j_i} \leq \pi(K_i) \leq l_{j_i}+\varphi_{j_i}+\lambda_{j_i}$, for $i = 1, \ldots, |K|$. The set J of all intervals j_i's for $i = 1, \ldots, |K|$ is clearly a dominating set for some $G \in \mathcal{F}_T$. Since $|D_G| \leq |J| = k(H)$, and $k(H) = \alpha(H)$, as H is a perfect graph, the claimed thesis follows. ❏

From this Lemma we immediately derive the following:

Theorem 5.10: Let T be a set of triples. Then $d(T)_{min} \leq \alpha(H)$.

A different upper bound can be obtained by relating the cardinality of a minimum dominating set of T to the cardinality of a minimum dominating set of a particular subset of triples, which we here define.

Definition 5.11: The *orderable subset of triples* T_p associated to the given set of triples T is obtained by removing from T all triples whose window contains another window of T.

Notice that the intersection graph H_p of the set of windows in T_p is an induced subgraph of H. Notice also that $T_p = T$ if T is orderable, and that every graph $G \in \mathcal{F}_{T_p}$ is a partial subgraph of H.

Theorem 5.12: Let T be a set of triples. Then $d(T)_{min} \leq d(T_p)_{min}$.

Proof: Consider a triple $t_u \in T \setminus T_p$. By definition of orderable subset of triples T_p, there exists (at least) a triple $t_v \in T_p$ whose window is properly contained into window u. If $(t_v, \varphi_v) \in D(T_p)$, for some φ_v, we are done. If $(t_v, \varphi_v) \notin D(T_p)$, consider a $(t_z, \varphi_z) \in D(T_p)$ which dominates t_v, that is either $l_v \leq l_z + \varphi_z \leq r_v$ or $l_v \leq l_z + \varphi_z + \lambda_z \leq r_v$ or both. Since $l_u \leq l_v$ and $r_v \leq r_u$, the triple t_z clearly dominates also t_u. ❑

The importance of this Theorem lays in the fact that problem Min $d(T_p)$ can be solved at optimality by Algorithm Greedy, in fact T_p is an orderable set of triples.

It can also be proved that

Theorem 5.13: Let T be a set of triples. Then $d(T_p)_{min} \leq \alpha(H)$.

In order to prove the theorem above we need the following

Lemma 5.14: Let T be a set of triples. Then $\alpha(H_p) = \alpha(H)$.

Proof: Let A be an independent set of H with maximum cardinality, that is $|A| = \alpha(H)$. Consider $A \subseteq V(H_p)$. Then A is an independent set for H_p, and of course $\alpha(H_p) = \alpha(H)$, as H_p is an induced subgraph of H. Consider $A \nsubseteq V(H_p)$ and be $x \in A \setminus V(H_p)$. Let $\mathrm{Adj}(v)$ denote the set of vertices adjacent to a vertex v in H and v itself. We claim that among the vertices adjacent to x in H there exists one, call it y, such that $y \in V(H_p)$ and $\mathrm{Adj}(y) \subseteq \mathrm{Adj}(x)$. This follows from the fact that there exists a window w_y properly contained into window w_x. Thus $A \setminus \{x\} \cup \{y\}$ keeps being a maximum cardinality independent set for H. ❑

Proof of Theorem 5.13: By Theorem 5.10, $d(T_p)_{min} \leq \alpha(H_p)$. By Lemma 5.14, $\alpha(H_p) = \alpha(H)$, and the claimed thesis follows. ❑

6. Conclusions and future work

In this paper we discussed of a set of intervals each allowed to be placed within a prescribed larger interval, called window. Depending on the exact placement of each interval within its window, the resulting intersection graph will have different values of some measures defined on it, like for example the size of a largest independent set or of a largest complete subgraph. Moreover, by considering the intersection graphs of all the sets of intervals obtained by varying their position in all possible feasible ways, we construct a family $\mathcal{F}_T$ of interval graphs.

We define an optimization problem on a family $\mathcal{F}_T$ of interval graphs. It consists in finding an interval graph of $\mathcal{F}_T$ on which a chosen classical graph measure has optimum value, over all the interval graphs of $\mathcal{F}_T$. In particular we chose, as measures, the size of a largest independent set, the size of a largest complete subgraph, and the size of a minimum dominating set.

The conducted study shows that the difficulty of an optimization problem defined on a set of shiftable intervals mainly resides in the selection from $\mathcal{F}_T$ of an interval graph with the required properties, and not on the evaluation of the chosen measure on the single interval graph. And in fact, in three cases an optimization problem defined on a set of shiftable intervals turns out to be NP–complete, while the evalutation of the corresponding measure on an interval graph has an $O(n \log n)$ computational complexity.

The general optimization problem we formulated in section 1, can be considered as a general framework which can be used to formulate many practical problems. A quite natural application is in the field of scheduling. As pointed out in the Proof of Theorem 4.1, the release date, due date, and processing times of a set of jobs fit into the triple set model in a natural way, as well as the schedules of the jobs correspond to placements of the intervals, and, for example, minimizing the number of identical machines is equivalent to minimizing the clique number, or minimizing the number of tardy jobs is equivalent to maximizing the independent set. Other problems can find a direct interpretation in terms of shiftable intervals. However, the study of a set of shiftable intervals by itself suggests many other interesting problems, as in the case when the chosen measure is the size of a dominating set.

The purpose of this paper is mainly introductory, hence only few of the possible features of the model have been dealt with. We think that, among the others, a couple of problems are of particular interest in this area. One is the recognition problem: given a set T of n triples, and an interval graph G defined on n vertices, which contains all the strong edges of H_T, and any subset of weak edges (possibly the empty one), does G belongs to $\mathcal{F}_T$? As noticed in section 2, the set of all interval graphs with the required properties contains $\mathcal{F}_T$, and in some case in a proper way, like in the discussed example. This means that the properties required for G, although quite restrictive, are still not sufficient enough for a precise description of $\mathcal{F}_T$. The second one consists in considering sets of circular shiftable intervals: intervals are allowed to be placed within windows which are arcs of a circumference (in other words, the problem is defined on a cylindric surface, instead of a two-dimensional plane one).

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