Abstract

In this paper new flexible collective transportation system are presented. The proposed system is suitable in the urban or interurban setting, and conjugate a traditional line transportation system with an on demand system. Several variants of the model are discussed, and for each variant a mathematical model is proposed. Solution approaches are also illustrated.

Demand Adaptive Systems: some proposals on flexible transit*

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1 Introduction

Traditional public transport systems are evolving towards more flexible organizations in order to capture additional demand and reduce operational costs. Such a trend calls for new models and tools to support the management of new services that make use of public facilities to meet individual needs.

A few solutions are already implemented and available in many urban centers; among them we mention the *limousine service*, *dial-a-ride service*, and *on-demand door-to-door transportation* for handicapped and elderly people [3, 7, 8].

Low demand transportation is a suitable candidate for implementing this kind of service: we speak of low demand transportation whenever the transportation system is not exploited up to its potential capacity, leaving room for alternative use of the resources involved. An example is the urban transport setting; this may concern buses traversing lines that serve areas with a low level of urbanization as well as buses operating during off-peak time slices or during holidays. In all cases the transportation service must be guaranteed although efficiency is also an issue.

This chapter aims to discuss a new transportation system that provides basic transportation, and at the same time it is able to attract additional passengers by allowing the individual user to induce detours in the vehicle routes through a new itinerary closer to the desired one. This system represents a good compromise between an expensive personalized service that precisely fulfills the individual request, such as a taxi ride would be, and the cheap alternative supplied by the traditional public transport which may not provide transportation exactly along the requested itinerary.

This chapter introduces a new transportation system, that we shall call Demand Adaptive System, which integrates traditional bus transportation on multiple lines and *on-demand* service. The suggested system is designed as follows. Let us consider a set of lines: each traversal of the line is described, as usually, in terms of a set of time-tabled trips. We shall call the stops in the original time table the *compulsory stops*. To introduce some flexibility into the vehicle routes, the vehicle is allowed to transit by each compulsory stop during a *time window*. Beside the compulsory stops, a set of stops to be activated on demand (*optional stops* hereafter) is available to the users. Between each pair of consecutive compulsory stops a set of optional stops is defined: this is the set of stops that can be visited during the trip from a compulsory stop to the next. Traveling times of the arcs of the physical network are known. A user issues a *service request* specifying a stop where to be picked up and a stop where to be dropped off. In the absence of requests involving optional stops, the vehicle travels along the shortest path on the

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network within each pair of consecutive compulsory stops. The acceptance of a request implies the rerouting of the vehicle for that part of route involving the optional stop(s) related to the request. Note that the detour may cause a delay of the transit time at the following stops.

The description of the system made above enlightens the differences between a Demand Adaptive System (DAS) and a Demand Responsive System (DRS), such as dial-a-ride, and so on. DAS adapts itself to attract as much demand as possible, but it operates within a conventional line transportation framework. Indeed, users who do not explicitly call for the service, but board and alight at compulsory stops only can also use DAS as a traditional bus service. On the other hand, DRS is an on demand and personalized service, which usually requires higher costs.

We can distinguish different Demand Adaptive System models depending on the policy used to deal with requests:

DAS1 : Requests *may be rejected*, if their acceptance cause infeasibilities, or are not economically worthy. If a request is accepted, users must be picked up and dropped off *exactly* at the stops they asked. Model DAS1 has been introduced in [4, 5].

DAS2 : Users are *always* picked up precisely at the requested stop, but they may be dropped off in the vicinity of the requested alighting stop (at the closest compulsory stop, in the worst case), if this cannot be included in the vehicle tour. For this inconvenience the service management pays a penalty to the users, that can be considered in terms of a discount on the travel fare.

DAS3 : Users are *always* served, but they can be picked up and/or dropped off at stops in the vicinity of the requested ones, and in these cases the service management pays a penalty.

The models' difficulties depend on the number of lines, or whether vehicles operate a single tour or multiple tours along the same line. Starting from the basic model, where a single vehicle runs once along a single line circuit, the model can be generalized by considering vehicles operating multiple tours, multiple intersecting lines, and users being allowed to board at one stop of a line and alight at a stop of another line, traveling on vehicles that connect in compulsory stops. In this latter case, synchronization features must be taken into account.

Two alternative policies can be adopted for request scheduling.

The first case processes the requests *off-line*. It selects a set of optional stops among the ones having been requested, such that the profit is maximized and the resulting tour can operated by the vehicle. Once the vehicle itinerary has been determined it cannot be modified.

The second scenario processes the requests *on-line*, taking into account the current position of the vehicles with respect to their schedules. The vehicles are rerouted, involving a reoptimization of their schedules and a feasibility check. In this case the problem is highly constrained since previously accepted requests cannot be discarded and scheduled time of boarding stops cannot be anticipated. In this context, time windows act also as a warranty of the quality of the service with respect to requests already accepted. Since rerouting involves the computation of shortest Hamiltonian paths and time response is crucial in the on-line context, exact algorithms are not suitable, while we have to rely upon fast heuristics such as insertion heuristics [9] or other approaches that we will introduce at the last paragraph.

We formalize the three different service models (DAS1, DAS2 and DAS3) as Mixed Integer Linear Programming problems, analyze their mathematical properties, and suggest heuristic procedures for their solution. Moreover, the models are extended considering a single line with one vehicle operating multiple tours. This study represents the starting point for analyzing the more general case of multiple lines. Finally the on-line problem is briefly discussed.

2 Notation and problems definition

We consider a line structured as a simple circuit, served by a single vehicle, starting and ending its tour at the same terminal. Along the circuit the vehicle passes by a sequence $H = \{f_1, f_2, \ldots, f_{n+1}\}$ of $n+1$ compulsory stops, where the terminal is the first (f_1) and the last ($f_{n+1} = f_1$) element of the sequence. For each stop f_h a time window $[a_h, b_h]$ is defined; the vehicle must leave f_h not before a_h and not later than b_h, but it may arrive there before a_h for $h = 1, \ldots, n$; b_{n+1} is the maximum trip completion time, and $a_1 = b_1$ is the starting time of the vehicle from the terminal.

A set F_h of optional stops is associated with each pair of consecutive compulsory stops $\langle f_h, f_{h+1} \rangle$. The vehicle passes by an optional stop only if a related boarding or alighting request has been issued. The sets F_h are mutually disjoint. Considering any pair $\langle f_h, f_{h+1} \rangle$ we can define a directed graph $G_h = (N_h, A_h)$, such that $N_h = F_h \bigcup \{f_h, f_{h+1}\}$ is the stops set and $A_h \subseteq N_h \times N_h$ is the set of arcs connecting the stops. In the following we shall refer to G_h as *segment* h. Finally, $G = (N, A)$ is the whole graph: $G = \bigcup_h G_h$. The travel time τ_{ij} and the travel cost c_{ij}, for each $(i, j) \in A$ of possible consecutive stops, either compulsory or optional, are given. Without loss of generality, we can suppose that the triangular inequalities hold.

Let us denote by P_h the set of paths in G_h from f_h to f_{h+1}. The vehicle itinerary in the segment h is a path $p \in P_h$, having travel time $\tau(p)$ and travel cost $c(p)$ given by the sum of the travel times and the costs of its arcs, respectively:

$$\tau(p) = \sum_{(i,j) \in p} \tau_{ij}; \qquad\qquad c(p) = \sum_{(i,j) \in p} c_{ij}. \tag{1}$$

Let t_h be the starting time from f_h; we assume that $t_1 = a_1$. The sequence of paths defined for each segment forms a *tour* q starting and ending at the terminal. Let us denote by $p_h \in P_h$ the path chosen in segment h; then, the arrival time at the end of the segment, that is at the stop f_{h+1}, is $t_h + \tau(p_h)$. The resulting tour q is *feasible* when:

(i) $t_{h+1} \geq t_h + \tau(p_h) \qquad h = 1, \ldots, n-1;$

(ii) $a_h \leq t_h \leq b_h \qquad h = 2, \ldots, n;$

(iii) $t_n + \tau(p_n) \leq b_{n+1}.$

Note that, since no feasible tour can contain a path whose travel time exceeds $b_{h+1} - a_h$, $h = 1, \ldots, n$, we can restrict P_h to the set of paths with travel time less than or equal to $b_{h+1} - a_h$. Let Q be the set of feasible tours; the *global cost* $c(q)$ of tour $q \in Q$ is given by the sum of the costs of the paths forming q.

Let $N(p_h)$, $h = 1, \ldots, n$, be the node set (i.e. the *served stops*) of path $p_h \in P_h$ of segment h, and let $N(q) = \bigcup_h N(p_h)$ be the set of all served stops.

Let us indicate by R the *request set*; the request $r \in R$ is defined as the pair $\langle s(r), d(r) \rangle$ of boarding and alighting stops; $h(s(r))$ and $h(d(r))$ represent the segments which the boarding stop $s(r)$ and the alighting stop $d(r)$ belong to, respectively. Let us assume that, for each request $r \in R$, $h(s(r)) < h(d(r))$ holds. The assumption that $s(r)$ and $d(r)$ can not belong to the same segment is quite realistic. Indeed, any two optional stops within the same segment are relatively close to each other. Because of this assumption no precedence constraints must hold between stops within the same segment, while precedence constraints regarding the pair stops of each request is implicitly handled by the sequencing of compulsory stops.

As far as model DAS1 is concerned, given a tour q, a request r is *satisfied* only if both the boarding and the alighting stops belong to $N(q)$; the subset $R(q) \subseteq R$ of the requests satisfied by tour $q \in Q$ is given by:

$$R(q) = \{r \in R : s(r), d(r) \in N(q)\}.$$

A *benefit* $u(r) \geq 0$ is associated with each request $r \in R$; the *global benefit* $u(q)$ of tour q is given by:

$$u(q) = \sum_{r \in R(q)} u(r).$$

Defining the *profit* as the difference between the benefit and the cost, the problem identified by model DAS1 is to find a tour $q^* \in Q$ of maximum profit:

$$u(q^*) - c(q^*) = \max\{u(q) - c(q) : q \in Q\}.$$

As far as models DAS2 and DAS3 are concerned, we notice that all requests are served. However, due to the time window constraints, not all users can be picked up or dropped off (depending on the service model under consideration) at the desired stops. Thus a global penalty measure must be associated with each feasible tour.

Given a request r defined by $\langle s(r), d(r) \rangle$, penalties $v'(r)$ and $v''(r)$ are associated with not serving stop $s(r)$ and $d(r)$, respectively. In this case the user is picked up or dropped off at some other stops in segments $h(s(r))$ and $h(d(r))$, at worst at the closest compulsory stop. Actually, the penalty should depend on which stop is selected for this purpose, therefore depending on the actual vehicle itinerary, but for sake of simplicity, we shall assume the penalty to be proportional to the distance between the missed stop and the closest compulsory stop. That is the service management offers a discount to the user as he/she where picked up or dropped off at a compulsory stop, even though he/she is allowed to take advantage of any other optional stop included in the actual tour.

Let us introduce the definition of *basic path* of segment h, denoted by $\bar{p}_h, h = 1, \ldots, n$, as the minimal path with respect to the request management policy. In the case of model DAS2, $\bar{p}_h$ is given by the least travel time path from f_h to f_{h+1} passing by all the optional stops corresponding to boarding requests in segment h, while, in case of model DAS1 and DAS3, the basic path $\bar{p}_h$ is the minimum travel time path from f_h to f_{h+1} without intermediate stops. Note that, in DAS3, $\bar{p}_h$ involves serving the users at the compulsory stops f_h or f_{h+1} rather than at the requested optional stops in segment h, while in model DAS1 $\bar{p}_h$ involves the rejection of all requests concerning optional stops in segment h. Moreover, let $\bar{q}$ denote the basic tour formed by the n basic paths. Note that in DAS2, such a tour might not exist, since even the basic tour passing by all boarding stops can violate the time window constraints. We will discuss this problem in section 4.

As far as models DAS2 and DAS3 are concerned, under the hypothesis of constant penalty made above, each optional stop f not belonging to the basic tour, once introduced in the tour, decreases the penalty by:

$$v(f) = \sum_{r: d(r) = f} v''(r) \tag{2}$$

in the case of DAS2, while in the case of DAS3 we have

$$v(f) = \sum_{r: s(r) = f} v'(r) + \sum_{r: d(r) = f} v''(r). \tag{3}$$

Given a path $p \in P_h$ we denote by $w(p)$ the *path net worth* with respect to the basic path $\bar{p}_h$:

$$w(p) = \sum_{f \in p, f \notin \bar{p}_h} v(f) - (c(p) - c(\bar{p}_h)) \tag{4}$$

Similarly we can define the *tour net worth* $w(q)$ as the sum of paths net worths:

$$w(q) = \sum_{p \in q} w(p).$$

Thus, regarding model DAS3, let U be the *net benefit* of the basic tour $\bar{q}$, that is the difference between the benefit of all requests and the sum of all penalties. While in model DAS2 the net benefit U is given by the difference between the benefit of all requests and the sum of all penalties due to alighting only stops. In models DAS2 and DAS3, the objective is to find the feasible tour q^* that minimizes the global penalty, that is the one maximizing the profit of the basic tour $\bar{q}$ plus the tour net worth of q^*:

$$U - c(\bar{q}) + \max\{w(q) : q \in Q\}.$$

3 DAS1: a request selection problem

As previously discussed, the main objective of service model DAS1 is to select the requests to be served, and find a maximum profit feasible tour. The mathematical model makes use of the following variables:

- $z_p^h = 1$ if path $p \in P_h$ is chosen, $z_p^h = 0$ otherwise, $\forall p \in P_h, h = 1, \ldots, n$;
- $y_r^s = 1$ if $s(r) \in N(p)$, where p is the chosen path ($z_p^h = 1$), $y_r^s = 0$ otherwise, $\forall r \in R$;
- $y_r^d = 1$ if $d(r) \in N(p)$, where p is the chosen path ($z_p^h = 1$), $y_r^d = 0$ otherwise, $\forall r \in R$;
- t_h = starting time of the vehicle from f_h, $h = 1, \ldots, n$, with $t_1 = a_1$.

Therefore, a request r is served if and only if y_r^s and y_r^d are equal to one.

$$(P1): \max \sum_{r \in R} u(r) y_r^s - \sum_{h=1}^{n} \sum_{p \in P_h} c(p) z_p^h$$

$$y_r^s \leq \sum_{p \in P_h} \delta_{s(r),p} z_p^h \qquad \forall r : s(r) \in N_h, h = 1, \ldots, n \tag{5}$$

$$y_r^d \leq \sum_{p \in P_h} \delta_{d(r),p} z_p^h \qquad \forall r : d(r) \in N_h, h = 1, \ldots, n \tag{6}$$

$$y_r^s = y_r^d \qquad \forall r \in R \tag{7}$$

$$\sum_{p \in P_h} z_p^h = 1 \qquad h = 1, \ldots, n \tag{8}$$

$$t_h + \sum_{p \in P_h} \tau(p) z_p^h \leq t_{h+1} \qquad h = 1, \ldots, n-1 \tag{9}$$

$$t_n + \sum_{p \in P_n} \tau(p) z_p^n \leq b_{n+1} \tag{10}$$

$$a_h \leq t_h \leq b_h \qquad h = 1, \ldots, n \tag{11}$$

$$y_r^s, y_r^d \in \{0, 1\} \qquad \forall r \in R$$

$$z_p^h \in \{0, 1\} \qquad \forall p \in P_h, h = 1, \ldots, n$$

where $\delta_{s(r),p} = 1$ if $s(r) \in N(p)$, $\delta_{s(r),p} = 0$ otherwise, $\forall p \in P_{h(s(r))}$, $\forall r \in R$, and $\delta_{d(r),p} = 1$ if $d(r) \in N(p)$, $\delta_{d(r),p} = 0$ otherwise, and $\forall p \in P_{h(s(r))}, \forall r \in R$.

Notice that variable y_r^s is equal to one if and only if the vehicle passes by stop $s(r)$ and request r is served, and, similarly, y_r^d is equal to one if and only if the vehicle passes by stop $d(r)$ and request r is served.

Constraints (5) and (6) link the choice of the path to the served requests; constraints (7) couple boarding and alighting stops for each request. Constraints (8) impose the selection of one path for each segment, while constraints (9), (10) and (11)state the requirement that the selected paths form a feasible tour.

Problem $(P1)$ is NP-Hard since a particular instance reduces to a TSP. Later on, we will discuss some methods to compute upper bounds and heuristic solutions for the problem.

3.1 DAS1+: a multiple tour service model

Let us now consider a service model where, instead of having one vehicle operating a single tour along the circuit line, K tours $\{q_1, \ldots, q_K\}$ of the same circuit line are performed. The multiple tours can be performed by a single vehicle consecutively, or by multiple vehicles. For the sake of simplicity, we focus on the case of a single vehicle, though the model can be immediately generalized to the case of multiple vehicles. In the multiple tour case, a request $r \in R$ is specified by a triplet $\langle s(r), d(r), i(r) \rangle$ where $i(r)$ is the *ideal service tour*; instead of simply discarding a request, the service management can decide to serve it in a tour different from the ideal one. Thus, the problem is actually a requests assignment to tours. In the following we give a mathematical formulation of the problem. Let $[a_h^i, b_h^i]$ be the time window at compulsory stop f_h during the i-th tour, $h = 1, \ldots, n$, $i = 1, \ldots, K$. Since the tours are sequentially operated, we can assume $b_{n+1}^i \leq a_1^{i+1}$ for $i = 1, \ldots, K-1$.

It is reasonable to assume that the benefit of a request depends on the tour it is assigned to: being zero if the request is served too late or too early with respect to $i(r)$, and decreasing as the service delay or earliness increases. Let $u^i(r)$ be the benefit associated with request r if served during the i-th tour, for each $r \in R$, $i = 1, \ldots, K$. Note that, oppositely to the case of the single tour, a request $r = \langle s(r), d(r), i(r) \rangle$ can have $h(d(r)) < h(s(r))$, that is a user can alight in a stop which is located before in the line with respect to the boarding stop. In this case the user is picked up during one tour and is dropped off during the successive tour. Let R' be the set of requests such that $h(s(r)) < h(d(r))$, and $R'' = R \backslash R'$.

The mathematical model is the following:

$$(P1+): \ \max \sum_{i=1}^{K} \sum_{r \in R} u^i(r) y_r^{s,i} - \sum_{i=1}^{K} \sum_{h=1}^{n} \sum_{p \in P_h} c(p) z_p^{h,i}$$

$$y_r^{s,i} \leq \sum_{p \in P_h} \delta_{s(r),p} z_p^{h,i} \qquad \forall r : s(r) \in N_h, h = 1, \ldots, n, i = 1, \ldots, K \tag{12}$$

$$y_r^{d,i} \leq \sum_{p \in P_h} \delta_{d(r),p} z_p^{h,i} \qquad \forall r : d(r) \in N_h, h = 1, \ldots, n, i = 1, \ldots, K \tag{13}$$

$$y_r^{s,i} = y_r^{d,i} \qquad \forall r \in R', i = 1, \ldots, K \tag{14}$$

$$y_r^{s,i} = y_r^{d,i+1} \qquad \forall r \in R'', i = 1, \ldots, K-1 \tag{15}$$

$$\sum_{i=1}^{K} y_r^{s,i} \leq 1, \qquad \forall r \in R \tag{16}$$

$$\sum_{p \in P_h} z_p^{h,i} = 1 \qquad h = 1, \ldots, n, i = 1, \ldots, K \tag{17}$$

$$t_h^i + \sum_{p \in P_h} \tau(p) z_p^{h,i} \leq t_{h+1}^i \qquad h = 1, \ldots, n-1, i = 1, \ldots, K \tag{18}$$

$$t_n^i + \sum_{p \in P_n} \tau(p) z_p^{n,i} \leq b_{n+1}^i \qquad i = 1, \ldots, K-1 \tag{19}$$

$$a_h^i \leq t_h^i \leq b_h^i \qquad h = 1, \ldots, n, i = 1, \ldots, K \tag{20}$$

$$y_r^{s,i}, y_r^{d,i} \in \{0,1\} \qquad \forall r \in R, i = 1, \ldots, K$$

$$z_p^{h,i} \in \{0,1\} \qquad \forall p \in P_h, h = 1, \ldots, n, i = 1, \ldots, K,$$

where variables t_h^i give the starting time from stop f_h in the i-th tour, variables $y_r^{s,i}$, $y_r^{d,i}$ are equal to one if the request r is served in the i-th tour, and variable $z_p^{h,i}$ is equal to one if path $p \in P_h$ belongs to the i-th tour. Constraints (16) state that all requests must be assigned to a tour at most.

If we consider the case where more vehicles operate on the same line, constraints (15) become:

$$y_r^{s,i} = y_r^{d,i+V} \qquad \forall r \in R'', i = 1, \ldots, K-V, \tag{21}$$

where V is the number of vehicles operating on the line. Obviously, as far as time windows are concerned, $b_{n+1}^i \leq a_1^{i+V}$, for $i = 1, \ldots, K-V$.

The solution approach to this problem can be similar to the one adopted for the single tour case, even though the size is much larger and other decompositions could be introduced.

4 DAS2: minimizing the penalty of alighting stops

Provided that all requests are served and all users are picked up at the desired stops, the main objective of service model DAS2 can be seen as defining a maximum net worth feasible tour. For the sake of simplicity we assume that there exists a basic tour $\bar{q}$, that is a tour passing by all boarding stops and fulfilling the time window requirements. Note that, in this model, P_h is the set of all paths in G_h passing by all nodes corresponding to boarding stops of segment h.

Let U be the net benefit of $\bar{q}$, that is:

$$U = \sum_{r \in R} u(r) - \sum_{r : d(r) \notin \bar{q}} v''(r).$$

where $v''(r)$ is the alighting penalty associated with request r. Let $\bar{F}_h$ be the set of optional stops not in the basic path $\bar{p}_h$ given by the path passing by all boarding stops of segment h:

$$\bar{F}_h = \{f \in F_h \backslash N_h(\bar{p}_h)\}.$$

For each optional stop $f \in \bar{F}_h$ we can compute the saving with respect to U resulting from the insertion of f in the path, as in (2), and for each feasible path $p \in P_h$ we can compute the *net worth* with respect to the basic path $\bar{p}_h$ as stated by (4).

Note that the net worth $w(\bar{p}_h)$ of the basic path $\bar{p}_h$ is equal to zero. Let P_h be the set of all paths going from f_h to f_{h+1}, passing by at least all boarding stops of segment h and fulfilling the time window constraints. The mathematical model of the problem of finding the feasible tour maximizing the net worth with respect to $\bar{q}$ is the following:

$$(P2) : \max \sum_{h=1}^{n} \sum_{p \in P_h} w(p) z_p^h$$

$$\sum_{p \in P_h} z_p^h = 1, \qquad h = 1, \ldots, n \tag{22}$$

$$t_h + \sum_{p \in P_h} \tau(p) z_p^h \leq t_{h+1}, \qquad h = 1, \ldots, n-1 \tag{23}$$

$$t_n + \sum_{p \in P_n} \tau(p) z_p^n \leq b_{n+1}, \tag{24}$$

$$a_h \leq t_h \leq b_h, \qquad h = 1, \ldots, n \tag{25}$$

$$z_p^h \in \{0, 1\} \qquad \forall p \in P_h, h = 1, \ldots, n.$$

Constraints (23), (24) and (25) state the feasibility of the departure times: the vehicle must leave f_h after it has arrived, and within the time window. It should be remarked that Problem $(P2)$ has a block structure, where each block corresponds to a segment. Note that the proposed model involves variables z_p^h only, while variables y_r^s and y_r^d have been omitted with respect to model $(P1)$. This is due to the fact that all requests are served and the selection of the stops to be served is implicit in the path choice and in the definition of $w(p)$. The simple formulation suggests a solution approach based on column generation methods, as discussed in section 6.2.

4.1 DAS2+: a multiple tour service model

As mentioned in the previous sections, the basic tour that passes by all boarding stops can be infeasible, that is it may violate the time windows constraints in some compulsory stops. In this circumstance, the service management may decide to discard some requests, reducing the problem to the one seen in the case of DAS1. Alternatively, we can think of a service system with multiple vehicles, or with a single vehicle making multiple tours. In this case, instead of discarding requests, the service management has to decide in which tour a requests has to be served. As in DAS1+, a different benefit can be associated with each request depending on the tour in which it is served. As in section 3.1, let $z_p^{h,i}$ be a variable equal to 1 when path p of segment h is selected during the i-th tour, and 0 otherwise, and let $y_r^{s,i}$ be a variable saying if request r is served during the i-th tour, that is the vehicle passes by stop $s(r)$ during the i-th tour. Moreover, $y_r^{d,i}$ is equal to one if the vehicle passes by stop $d(r)$ in the i-th tour and request r is served. The mathematical model is:

$$(P2+) \quad : \quad \max \sum_{i=1}^{K} \sum_{r \in R} u^i(r) y_r^{s,i} - \sum_{r \in R} v_r^{''} (1 - \sum_{i=1}^{K} y_r^{d,i}) + \\ - \sum_{i=1}^{K} \sum_{h=1}^{n} \sum_{p \in P_h} c(p) z_p^{h,i}$$

$$y_r^{s,i} \leq \sum_{p \in P_h} \delta_{s(r),p} z_p^{h,i} \qquad \forall r : s(r) \in N_h, h = 1, \dots, n, i = 1, \dots, K \tag{26}$$

$$y_r^{d,i} \leq \sum_{p \in P_h} \delta_{d(r),p} z_p^{h,i} \qquad \forall r : d(r) \in N_h, h = 1, \dots, n, i = 1, \dots, K \tag{27}$$

$$y_r^{s,i} \geq y_r^{d,i} \qquad \forall r \in R', i = 1, \dots, K \tag{28}$$

$$y_r^{s,i} \geq y_r^{d,i+1} \qquad \forall r \in R'', i = 1, \dots, K-1 \tag{29}$$

$$\sum_{i=1}^{K} y_r^{s,i} = 1, \qquad \forall r \in R \tag{30}$$

$$\sum_{i=1}^{K} y_r^{d,i} \leq 1, \qquad \forall r \in R \tag{31}$$

$$\sum_{p \in P_h} z_p^{h,i} = 1 \qquad h = 1, \dots, n, i = 1, \dots, K \tag{32}$$

$$t_h^i + \sum_{p \in P_h} \tau(p) z_p^{h,i} \leq t_{h+1}^i \qquad h = 1, \dots, n-1, i = 1, \dots, K \tag{33}$$

$$t_n^i + \sum_{p \in P_n} \tau(p) z_p^{n,i} \leq b_{n+1}^i \qquad i = 1, \dots, K-1 \tag{34}$$

$$a_h^i \leq t_h^i \leq b_h^i \qquad h = 1, \dots, n, i = 1, \dots, K \tag{35}$$

$$y_r^{s,i}, y_r^{d,i} \in \{0, 1\} \qquad \forall r \in R, i = 1, \dots, K$$

$$z_p^{h,i} \in \{0, 1\} \qquad \forall p \in P_h, h = 1, \dots, n, i = 1, \dots, K.$$

Note that constraints (31) are redundant being implied by constraints (28), (29) and (30).

The assignment of requests to tours is explicitly specified by way of variables $y_r^{s,i}$. In the objective function the contribution of the benefit of a request r depends on the tour the request is assigned to (stated by variables $y_r^{s,i}$). Moreover, the penalty in case of displacement (depending on variables $y_r^{d,i}$) and the cost of the selected paths have to be subtracted. Smilarly to DAS1+, the model can be generalized in order to deal with multiple vehicles.

5 DAS3: minimizing the penalty of boarding and alighting stops

In service model DAS3 all requests are served, but users, instead of being picked up and/or dropped off at the desired stops, can be picked up and/or dropped off at alternative stops. This opportunity allows the service management to determine a feasible tour in any demand condition. As in the case of DAS2 the inconveniences caused to the users will be payed in terms of penalties (discounts on the transit fare). The approach is similar to that of DAS2, though simplified. Let us redefine some concepts taking into account the additional degree of freedom introduced in the boarding stops. For each segment h the basic path $\bar{p}_h$ goes straight from f_h to f_{h+1}, thus the set of optional stops to be considered is:

$$\bar{F}_h = \{f \in F_h : \exists r \in R, f = d(r) \text{ or } f = s(r)\}.$$

For each optional stop $f \in \bar{F}_h$ we can compute the saving with respect to U as stated by (3); as in the case of DAS2, for each feasible path $p \in P_h$ we can compute the *net worth* with respect to the basic path $\bar{p}_h$ as in (4).

The mathematical model of the problem of finding the feasible tour maximizing the net worth with respect to $\bar{q}$ is the following, and is exactly the same of $(P2)$ except for the definition of $w(p)$, $\bar{F}_h$, and P_h. In particular, the set of feasible paths P_h is, as in DAS1, the set of paths going from f_h to f_{h+1} and it is larger than in model DAS2, since paths are not obliged to pass by all boarding stops of segment h.

$$(P3) : \max \sum_{h}^{n} \sum_{p \in P_h} w(p) z_p^h$$

$$\sum_{p \in P_h} z_p^h = 1, \qquad h = 1, \ldots, n \tag{36}$$

$$t_h + \sum_{p \in P_h} \tau(p) z_p^h \le t_{h+1}, \qquad h = 1, \ldots, n-1 \tag{37}$$

$$t_n + \sum_{p \in P_n} \tau(p) z_p^n \le b_{n+1}, \tag{38}$$

$$a_h \le t_h \le b_h, \qquad h = 1, \ldots, n \tag{39}$$

$$z_p^h \in \{0, 1\} \qquad \forall p \in P_h, h = 1, \ldots, n.$$

Note that in the case of DAS3 there always exists a feasible solution, for example $\bar{q}$.

5.1 DAS3+: a multiple tour service model

As in the case of DAS2, model DAS3 can be extended to comply with multiple tours. Note that, by contrast with DAS2+, even if a request is boarded during the i-th tour, both $y_r^{d,i}$ and $y_r^{s,i}$ are equal to 0 whenever the user is not boarded and alighted at the requested stops. Therefore, a new set of variables is needed to denote the requests assignment to tours: let y_r^i be equal to 1 if request r is boarded during the i-th tour, and 0 otherwise. The mathematical model is the following:

$$(P3+) \quad : \quad \max \sum_{i=1}^{K} \sum_{r \in R} u^i(r) y_r^i - \sum_{r \in R} v_r^{'} (1 - \sum_{i=1}^{K} y_r^{s,i}) - \sum_{r \in R} v_r^{''} (1 - \sum_{i=1}^{K} y_r^{d,i})$$
$$- \sum_{i=1}^{K} \sum_{h=1}^{n} \sum_{p \in P_h} c(p) z_p^{h,i}$$

$$y_r^{s,i} \leq \sum_{p \in P_h} \delta_{s(r),p} z_p^{h,i} \qquad \forall r : s(r) \in N_h, h = 1, \ldots, n, i = 1, \ldots, K \tag{40}$$

$$y_r^{d,i} \leq \sum_{p \in P_h} \delta_{d(r),p} z_p^{h,i} \qquad \forall r : d(r) \in N_h, h = 1, \ldots, n, i = 1, \ldots, K \tag{41}$$

$$\sum_{i=1}^{K} y_r^{s,i} \leq 1, \qquad \forall r \in R \tag{42}$$

$$\sum_{i=1}^{K} y_r^{d,i} \leq 1, \qquad \forall r \in R \tag{43}$$

$$\sum_{i=1}^{K} y_r^{i} = 1, \qquad \forall r \in R \tag{44}$$

$$y_r^{s,i} \leq y_r^i \qquad \forall r \in R, i = 1, \ldots, K \tag{45}$$

$$y_r^{d,i} \leq y_r^i \qquad \forall r \in R', i = 1, \ldots, K \tag{46}$$

$$y_r^{d,i+1} \leq y_r^i \qquad \forall r \in R'', i = 1, \ldots, K-1 \tag{47}$$

$$\sum_{p \in P_h} z_p^{h,i} = 1 \qquad h = 1, \ldots, n, i = 1, \ldots, K \tag{48}$$

$$t_h^i + \sum_{p \in P_h} \tau(p) z_p^{h,i} \leq t_{h+1}^i \qquad h = 1, \ldots, n-1, i = 1, \ldots, K \tag{49}$$

$$t_n^i + \sum_{p \in P_h} \tau(p) z_p^{n,i} \leq b_{n+1}^i \qquad i = 1, \ldots, K-1 \tag{50}$$

$$a_h^i \leq t_h^i \leq b_h^i \qquad h = 1, \ldots, n, i = 1, \ldots, K \tag{51}$$

$$y_r^{s,i}, y_r^{d,i}, y_r^i \in \{0, 1\} \qquad \forall r \in R, i = 1, \ldots, K$$

$$z_p^{h,i} \in \{0, 1\} \qquad \forall p \in P_h, h = 1, \ldots, n, i = 1, \ldots, K.$$

for the sake of clarity, we explicitly formulate constraints (42) and (43), even though they are redundant, since implied by (44), (45), (46) and (47).

As for DAS1+and DAS2+, model DAS3+ can be generalized to deal with multiple vehicles.

Model DAS3+ can be extended in order to guarantee a good service level as, for example, by introducing the rule according to which a passenger either is displaced in space or in time. In the first case a request r is not served at the requested stops, but it must be served during the ideal tour $i(r)$; in the other case a request is not served during tour $i(r)$, but it must be served at the requested stops. This condition is enforced by the following set of constraints:

$$\sum_{i \neq i(r)} y_r^{s,i} + y_r^{i(r)} = 1 \qquad \forall r \in R, i = 1, \ldots, K \tag{52}$$

$$\sum_{i \neq i(r)} y_r^{d,i} + y_r^{i(r)} = 1 \qquad \forall r \in R', i = 1, \ldots, K \tag{53}$$

$$\sum_{i \neq i(r)} y_r^{d,i+1} + y_r^{i(r)} = 1 \qquad \forall r \in R'', i = 1, \ldots, K-1. \tag{54}$$

Note that the same rule can be introduced in model DAS2+ also, by adding constraints (53) and (54) where $y_r^{i(r)}$ is replaced by $y_r^{s,i(r)}$.

6 Solution approaches

In this section we propose three possible approaches to the models described so far. The first is a Lagrangian Relaxation, which is applicable to $(P1)$ but it can be extended to $(P1+)$, $(P2+)$, and $(P3+)$. The second approach is a Column Generation, which is suited for $(P2)$, and $(P3)$, where request selection variables are not present. These approaches, beside an upper bound value, yield other information which can be exploited in the construction of a heuristic solution. A possible heuristic approach exploiting this information is discussed too.

6.1 Lagrangian Relaxation

Let us illustrate the case of DAS1.

The Lagrangian relaxation of constraints (7), (9) and (10) with multipliers λ and μ, respectively, yields a set of separable subproblems, one for each segment h, of the form:

$$(P(h)_{\lambda,\mu}) : \max \sum_{r:s(r) \in N_h} (u(r) - \lambda_r) y_r^s + \sum_{r:d(r) \in N_h} \lambda_r y_r^d$$

$$-(\mu_h - \mu_{h-1}) t_h - \sum_{p \in P_h} (c(p) + \mu_h \tau(p)) z_p^h$$

$$y_r^s \leq \sum_{p \in P_h} \delta_{s(r),p} z_p^h \qquad \forall r : s(r) \in N_h$$

$$y_r^d \leq \sum_{p \in P_h} \delta_{d(r),p} z_p^h \qquad \forall r : d(r) \in N_h$$

$$\sum_{p \in P_h} z_p^h = 1$$

$$a_h \leq t_h \leq b_h$$

$$y_r^s \in \{0, 1\} \qquad \forall r : s(r) \in N_h$$

$$y_r^d \in \{0, 1\} \qquad \forall r : d(r) \in N_h$$

$$z_p^h \in \{0, 1\} \qquad \forall p \in P_h$$

Each subproblem $(P(h)_{\lambda,\mu})$, beside being of smaller size with respect to the global problem $(P1)$, is also quite affordable to solve. In fact the request benefits (including the effect of multipliers λ) can be included in a node weight of graph G_h, and the cost of the links (including the effect of multipliers μ) can be included in an arc weight of the same graph. Thus the problem is a maximum path problem with a constraint on the travel time. It should be noted that in graph G_h, with the above weight definitions, positive cycles may occur; however, due to the time windows constraints, the problem is bounded and can be efficiently solved as described in [2].

The Lagrangian Dual is solved by way of a bundle algorithm [1] which provides an upper bound to the problem, a collection of feasible paths for each segment, and a set of optimal multipliers.

6.2 A Column Generation approach

Problems $(P2)$ and $(P3)$ can be naturally approached through Column Generation. Let us briefly describe the method in the general case, since the algorithm can be easily specialized considering the different definitions of P_h and $w(p)$ given for DAS2 and DAS3. The master problem $(\bar{P})$ is the LP relaxation of $(P2)$ or $(P3)$ which considers paths in $P'_h \subseteq P_h, h = 1, \ldots, n$, only. According to the Column Generation method, $(\bar{P})$ is solved to optimality. Then, paths in $P_h \backslash P'_h$ are searched such that, if considered, would improve the current solution. Initially P'_h may contain $\bar{p}_h$ only, or a set of feasible paths heuristically determined. Due to the block structure, the search of a path to enter the problem can be decomposed into subproblems, one for each segment. The master problem $(\bar{P})$ is the following:

$$(\bar{P}) : \max \sum_{h=1}^{n} \sum_{p \in P'_h} w(p) z_p^h$$

$$\sum_{p \in P'_h} z_p^h = 1, \qquad h = 1, \ldots, n \tag{55}$$

$$t_h + \sum_{p \in P'_h} \tau(p) z_p^h \leq t_{h+1}, \qquad h = 1, \ldots, n-1 \tag{56}$$

$$t_n + \sum_{p \in P'_h} \tau(p) z_p^n \leq b_{n+1}, \tag{57}$$

$$a_h \leq t_h \leq b_h, \qquad h = 1, \ldots, n$$

$$0 \leq z_p^h \qquad \forall p \in P'_h, h = 1, \ldots, n.$$

Let π_h, σ_h be the optimal dual variables of problem $(\bar{P})$ associated with constraints (55), (56) and (57), respectively. The column generation phase seeks a feasible path in any segment whose reduced cost with respect to π_h, σ_h is greater than zero. If such a path p exists for some h, then the current solution of $(\bar{P})$ can be improved by adding p to P'_h; when such a path does not exist for each h, the current solution is optimal for the LP relaxation of $(P2)$ or $(P3)$. Let us briefly analyze the problem of searching a positive reduced cost path in segment h.

The reduced cost of a path $p \in P_h$ is:

$$\bar{w}(p) = w(p) - \pi_h - \sigma_h \tau(p), \qquad \forall p \in P_h, h = 1, \ldots, n;$$

by applying the definition of path net worth (4), path travel time, and path cost (1), we get:

$$\bar{w}(p) = -\pi_h - \sum_{(i,j)\in p} (c_{ij} + \sigma_h \tau_{ij}) + \sum_{f\in \bar{F}_h \cap p} v(f) + c(\bar{p}_h).$$

The search of a feasible path with positive reduced cost can be accomplished by looking for the longest feasible path from f_h to f_{h+1} in G_h, where arcs have weight $-(c_{ij} + \sigma_h \tau_{ij})$ and nodes have weight $v(f)$ if $f \in \bar{F}_h$, and zero otherwise. In the case of DAS2 the path has to pass by all the boarding nodes, while for DAS3 this restriction does not hold. This problem is easily solved by means of longest path algorithms; actually, paths may contain positive cycles, but the problem reduces to a classical maximum length path on a suitable space-time network [6].

6.3 Heuristic approaches

Since all the problems considered so far can be viewed as a particular *Vehicle Routing Problems*, classical heuristics are suitable to approach them. In particular, we can devise several "insertion" heuristics [9], where a tour is built from the basic one by iteratively inserting new optional stops in order to satisfy more requests or reduce penalties. Insertion criteria may vary depending on the transportation model.

Beside the aforementioned heuristics, another kind of appoach can be conceived. The proposed algorithm exploits the particular structure of the problem and the information yielded by the Lagrangian Relaxation or the Column Generation. in particular both methods yield a set of "promising" paths. However, such a set can be suitably integrated by other methods. The algorithm assembles paths in a greedy fashion, selecting them from the set of promising paths, trying to build a maximum profit tour.

The approach, with minor modifications, is suitable for all the three models so far introduced (DAS1, DAS3 and DAS3), and can be easily adapted to deal with the multiple tour cases.

The algorithm can be summarized as follows. Consider a set of paths $P' = \cup_h P'_h$, where P'_h corresponds to the set of promising paths of segment h. Let the basic tour $\bar{q}$ made by the n basic paths $\bar{p}_h$ be the starting solution. The algorithm iteratively improves the value of the current feasible tour q by swapping basic paths in q with paths in P' according to a greedy criterion. At each step, the *most promising* path with respect to the current tour q is selected and removed from P'; the meaning of most promising path depends on the kind of transportation model, as we will see later on. Let h be the segment of such a path, then the algorithm *checks the feasibility* of the new tour obtained from q by swapping the selected path with $\bar{p}_h$. If feasibile, P' is updated by removing all paths in P'_h, so that the selected path will belong to the final solution. The algorithm stops when P' becomes empty.

Two issues must be addressed in order to implement this procedure: how to check the feasibility of the tour yielded by the swapping, and the criteria according to which paths are selected.

Regarding the feasibility check of the new tour q, note that the path relative to segment h, denoted by $p_h(q)$, can be thoroughly characterized by the following information: the earliest departure time (EDT), the latest departure time (LDT), the earliest arrival time (EAT), and the latest arrival time (LAT), which can be iteratively computed according to:

$$\text{EDT}(p_h(q)) = \begin{cases} a_1 & \text{if } h = 1, \\ \max\{a_h, \text{EDT}(p_{h-1}(q)) + \tau(p_{h-1}(q))\} & \text{if } h = 2, \ldots, n; \end{cases}$$

$$\text{LDT}(p_h(q)) = \begin{cases} \min\{b_n, b_{n+1} - \tau(p_n(q))\}; & \text{if } h = n; \\ \min\{b_h, \text{LDT}(p_{h+1}(q)) - \tau(p_h(q))\} & \text{if } h = n-1, \ldots, 1, \end{cases}$$

$$\text{EAT}(p_h(q)) = \text{EDT}(p_h(q)) + \tau(p_h(q)) \qquad h = 1, \ldots, n; \tag{58}$$

$$\text{LAT}(p_n(q)) = \text{LDT}(p_h(q)) + \tau(p_h(q)) \qquad h = 1, \ldots, n. \tag{59}$$

Let τ_h be the maximum travel time for paths in P'_h in order to qualify for swapping.

$$\tau_h = \text{LDT}(p_{h+1}(q)) - \max\{\text{EAT}(p_{h-1}(q)), a_h\}.$$

Therefore, a path $p \in P'_h$ can be swapped with $\bar{p}_h$ only if $\tau(p) \leq \tau_h$

Regarding the path selection criteria, paths in P' may be ranked according to a score $s(p)$ that is a measure of the benefits coming from including path p in the final solution, with respect to the current solution q.

Relatively to models DAS2 and DAS3, the contribution of path p is given exactly by the *net worth* $w(p)$ which, by definition, gives the gain obtained by swapping the basic path with p. Note that the score does not depend on the other $n-1$ paths of the tour.

On the other hand, in the case of model DAS1, the score of a path does depend on the other paths in the tour, in particular the benefit of a request whose alighting node is in p contributes to the value of the solution only provided that the tour passes by the boarding node (and viceversa). Therefore, unlike the previous case, the score of path p varies according to the current solution q.

Let us restate the profit of tour q in terms of pairs of paths. Let $\theta(p, p')$, with p and p' in different segments, be the benefit of all requests having the boarding node in p and the alighting node in p' or vice versa. Then

$$u(q) - c(q) = \sum_{h=1}^{n-1} \sum_{h'=h+1}^{n} \theta(p, p') - \sum_{h=1}^{n} c(p_h(q)) \tag{60}$$

but, with respect to the current solution q, only the paths resulting from a swap are known to belong to the final solution.

Given $p \in P'_h$ the score $s(p)$ can be computed as follows. For each segment $l \neq h$ for which a path has not been already selected, the path p_l such that $\tau(p_l) \leq \tau_l$ and which maximizes $\theta(p, p_l) - c(p_l)$ is considered. The score $s(p)$ corresponds to the profit of the tour given by the selected paths, path p, and all paths p_l according to (60). Note that this tour may not be feasible, therefore the score is an upper bound of the profit of any tour containing p and the paths already selected.

Note that, after any successful swapping of $\bar{p}_h$ with $p \in P'_h$, $\text{EDT}(p_{h'}(q))$ must be updated for all $h' \geq h$, $\text{LDT}(p_{h'}(q))$ must be updated for all $h' \leq h$, while EAT and LAT are recomputed according to (58) and (59), respectively. As a consequence, for each segment h for which a path has not been already selected, τ_h may have been decreased, thus reducing the cardinality of set P'_h. In case of model DAS1 more updating is necessary since some paths p_l may not belong to P'_h any more, and the scores of some nodes may have to change.

7 On-line approaches

The ideal and more realistic service in low demand conditions is an *on-line* service, where requests can arrive also during the service duty of the vehicle. We may assume that some basic rules characterize an on-line service:

- once a request has been accepted, it cannot be discarded;

- once a service has been determined for a request (pick up stop, drop off stop, tour which it has assigned to), it cannot be modified or in general worsened;
- the service request reservations must be processed in a first–come–first–serve fashion;
- the service request reservation must be confirmed (i.e. accepted, rejected or modified proposing alternative pick up and/or drop off stops, and tour of assignment) in *real time*.

Actually, these rules simplify the optimization problem that must be solved each time a request is issued, since the number of feasible alternatives to the current solution is much smaller with respect to the off-line case. Let us briefly discuss how the solution approaches to DAS1, DAS2 and DAS3 change under the on-line perspective. Assume that the current established feasible tour is q and a new service request r arrives asking to be served starting from the current instant of time, and the vehicle has not passed by the desired boarding stop.

In the DAS1 framework, the problem is to decide whether $s(r)$ and $d(r)$ can be included in q, if not already present, maintaining the feasibility. There are two possible approaches:

- recomputing a new optimal tour q' including $s(r)$ and $d(r)$, and if a feasible solution is found, r is accepted and q' is the new current tour; the request is rejected, otherwise.
- heuristically evaluating if $s(r)$ and $d(r)$ can be inserted into q. If the answer is negative, the request is discarded even though there is not theoretical evidence that no feasible tour including r exists. If the answer is positive, the current tour can be refined after having confirmed the acceptance of the request.

Another possible way to implement a real time answer to service requests, is to store a set of good feasible solutions in a data base and efficiently retrieve the best among the available ones.

In the DAS2 case, the problem is slightly more delicate as it has to be decided whether the insertion of $s(r)$ into q is feasible, and whether it is more profitable to insert $d(r)$ into q or to pay a penalty $v''(r)$. Also in this case there are two possible approaches. The exact one computes the optimal tour, the heuristical one evaluates the feasibility of the possible insertion of $s(r)$ and the feasibility/profitability of the possible insertion of $d(r)$. In the heuristical approach the evaluation can be followed by a refining phase where the optimal tour is recomputed exactly, or with more accuracy.

In the DAS3 case, the problem is simply to evaluate the profitability of making a detour for boarding and/or alighting request r. This can be done by applying the same ideas seen in the case of DAS2.

The DAS1+, DAS2+, and DAS3+ systems in the on line case can be approached very simply. Provided that it is not reasonable to serve a request with more than one tour of delay with respect to the ideal one, it is sufficient to *unroll* the circuit line twice. That is the planning horizon is of two circuits instead of one. This unrolling can be dynamically updated as segments are visited by the vehicle.

7.1 Hybrid on-line solutions

The on-line approach requires a quite heavy technological support: vehicle monitoring systems, communication system between the service management center and the vehicles, and between the stops and the service management center. Often, in a preliminary testing phase, a cheaper system with less technological requirements is highly desirable, even though it can not implement all the features of the on-line service described above. Let us consider a possible hybrid on-line system. In such a system, requests can arrive during the vehicles are operating, but the itinerary of each vehicle is determined on the basis of the data available at the beginning of the tour, and can not be modified while the vehicle is running. In practice, the service management has to solve a sequence of single tour problems, (($P1$) or ($P2$) or ($P3$) according to the system selected) just before the vehicle leaves the terminal. The problems are solved using the service requests collected while the vehicle is performing the current tour and those that could not be previously served. In order to avoid a request being indefinitely hold on, a suitable priority mechanism has to be introduced, such that in the solution of a single tour problem older requests have better chances to be served.

8 Conclusions and future work

In this chapter we presented a new transit model identified as Demand Adaptive System. The system can be seen as a hybrid solution between a conventional line transit and an on-demand personalized transportation. This solution should combine the advantages of the two systems: the low costs and the reliability of a line transit on one side, and the flexibility and the capability of attracting users of a personalized transportation service on the other side. In the DAS framework we proposed several transportation models that are characterized by the different ways of managing the service requests. For each model we proposed a mathematical formulation and solution techniques in the case of single line systems.

A transportation system like the one proposed has several interesting aspects, especially nowadays. The transportation companies can implement the DAS to improve the efficiency and maintain or even enhance the service level. This point appears more and more critical.

Techological improvements should require:

- network vehicle monitoring system,
- communication systems between vehicles, users and managing center,
- efficient and clear information system for the users.

Indeed many of these systems have been already implemented and are currently adopted by many transit companies.

However, if these system are not already implemented, less requiring solutions can be also considered. In fact the hybrid or the off line models do not require an exact location of the vehicles on the network, and can rely on the telephone network (fix or cellular) for communications between the users, the vehicles and the managing center.

From the mathematical and algorithmic viewpoint the proposed model put a new light on how approach a transportation problem. We believe that this kind of transportation systems deserve to be studied and tested in practice in the near future.

The preliminary computational results regarding the off line models [4] are encouraging. Some test problems have been generated in order to reproduce possible urban settings with up to 10 segments, a number of stops ranging from 25 to 140, and a number of requests ranging from 40 to 190. The proposed relaxation gives a solution in a reasonable time (order of minutes of a medium sized workstation), and its quality is acceptable even for large instances. The value of the relaxation favorably compares with that of the trivial linear programming relaxation obtained by relaxing the integrality constraints and by solving the problem by means of standard LP codes. In fact the linear relaxation is usually quite poor, while the Lagrangean Relaxation is always more tight. The heuristic algorithms proposed produced the optimal solution in a very short time for all the small instances, where it has been possible to compute the optimal solution with a branch and bound. For larger instances the value of the heuristic solution is not far from the upper bound given by the relaxation. The implementation of all the proposed algorithms does not require any particular computational platform (an up to date PC is sufficient) nor any particular commercial software.

Model extensions should deal with a system with multiple DAS. When the lines share some stops (both compulsory and optional) two levels of problems arise. One passenger may ask to be transported from an optional stop of one line to an optional stop of another line. First of all, the system has to decide where to allow the passenger to transfer from one line to another. Moreover, it is necessary to enforce the synchronization of the vehicles involved by the transfers. All those ideas may introduce some challenging aspects from the mathematical viewpoint, especially when the off-line planning is considered. These aspects are currently under study.

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