A polynomially solvable class of Quadratic Semi-Assignment Problems[1]

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Abstract: The Quadratic Semi-Assignment Problem (QSAP) models a large variety of practical applications. In the present note we will consider a particular class of QSAP that can be solved by determining the maximum cost flow on a network. This class of problems arises in schedule synchronization and in transportation.

Keywords: Programming Quadratic, Timetabling.

1. Introduction

The Quadratic Semi-Assignment Problem (QSAP) has an important role in modelling many practical applications. For example clustering and partitioning problems [7], assignment of researchers to departments [6], some scheduling problems [4]; moreover Weighted Max 2-Sat can be reduced trivially to the QSAP. In general the QSAP is NP-hard [11], and in practice it is very difficult to provide the optimal solution, even for problems of small size [8]. In [9, 10] a class of polynomially solvable QSAP's is presented and it is exploited to devise lower bounds for the general case.

In the present note, another class of polynomially solvable QSAP's will be considered. This class arises for example from Schedule Synchronization Problem (SSP): consider a set of interconnected flights, a communication network and a departure time window for each flight, and let the estimated passenger flow at the

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interconnections be known; the Schedule Synchronization Problem consists of selecting the departure time for each flight within the given window, in order to minimize the total waiting time spent for connecting. Often in the literature, the various aspects of this problem have been considered separately. For a brief review refer to [12]. Recently global approaches to the problem have been proposed in the mass transit context ([5, 12]). The common and more natural approach is the use of a Quadratic Semi-Assignment model. We will see that, under quite reasonable hypotheses, this kind of problem can be solved in polynomial time. This class of easy QSAP's can arise also in the planning and improvement of regional mass transit with resource constraints (see [3]).

In the following we will present the problem formulation in the most general form giving some references to the SSP. Moreover we will devise a transformation that allows to solve the problem by means of a standard network flow algorithm.

2. Problem formulation

Consider a convex bipartite graph $G = (U, V, A)$, where the set of arcs A is a subset of the Cartesian product of the origin node set U and the destination node set V. Assuming that V is totally ordered, we say that G is *convex* when for each $i \in U$, h, $k \in V$, $h \le k$, the following implication holds:

$$(i,h) \in A \text{ and } (i,k) \in A \quad \Rightarrow \quad (i,l) \in A \qquad \forall l \in V,\ h \le l \le k.$$

This means that, for each origin i, the set of arcs incident with i can be described by the a pair of nodes a_i and b_i of V, where a_i and b_i represent the first and the last destination adjacent to origin i. Further on, we will assume that the origin and destination nodes are represented by the integers from 1 to $|U|$ and $|V|$, respectively. The problem of recognizing convex bipartite graphs has been dealt with in [2], where a linear time algorithm is proposed.

Consider the following QSAP defined on a convex bipartite graph:

$$\min \sum_{ij\in U} \sum_{hk\in V} q_{ihjk}\, x_{ih} x_{jk}$$

$$\sum_{h\in\{a_i..b_i\}} x_{ih} = 1, \qquad \forall i\in U, \qquad (2.1)$$

$$x_{ih} \in\{0,\ 1\} \qquad \forall i\in U,\ \forall h\in V.$$

Obviously, decision variable x_{ih}=1 if and only if origin i is assigned to destination h, and zero otherwise. The constraints of the problem state that each origin must be assigned to exactly one destination.

Let the quadratic coefficients be defined as follows:

$$q_{ihjk} = \begin{cases} Mp_{ij} & \text{if } k\text{-}h<r_{ij}, \\ p_{ij}(k\text{-}h\text{-}r_{ij}) & \text{otherwise,} \end{cases} \qquad (2.2)$$

where M is a suitably large scalar (for example $M>|U|^2|V| \max\{p_{ij}: i,j\in U\}$), and r_{ij} are integer coefficients, for each $i, j \in U$.

Problem (2.1), with costs defined by (2.2), has an immediate interpretation in terms of schedule synchronization. The origins represent the flights and the destinations represent the possible departure times; hence $\{a_i..b_i\}$ is the time window of feasible departure times for flight i. Coefficient p_{ij} represents the estimated amount of passengers that connect flight i with flight j, while coefficient r_{ij} denotes the travel time of flight i (note that, actually in the case of SSP, r_{ij} does not depend on j). If two flights are not in connection (i.e. i arrives after that j has left, that is $h+r_{ij}>k$) and $p_{ij}>0$ then a penalty results in the objective function, otherwise there is a contribution equal to the waiting time multiplied by the number of passengers.

Let us define the following transformation of the problem; we introduce new variables and substitute them to the 0-1 variables:

$$\pi_i = \sum_{h\in\{a_i..b_i\}} hx_{ih} \qquad \forall i\in U.$$

In practice, in the case of SSP, variable π_i gives the departure time of flight i.

If we express the penalties of the objective function as explicit constraints, we get the following formulation:

$$
\begin{aligned}
&\min \sum_{ij} p_{ij}(\pi_j-\pi_i-r_{ij}) \\
&\pi_j-\pi_i \geq r_{ij} && \forall i,j \in U \text{ s. t. } p_{ij}>0, \\
&a_i \leq \pi_i \leq b_i && \forall i \in U, \qquad (2.3) \\
&\pi_i \in \mathbf{Z}_+.
\end{aligned}
$$

It is easy to see that problem (2.3) has solution if and only if the optimal solution of problem (2.1) does not induce any penalty, that is the optimal solution value is less than M, and in this case the two problems are equivalent.

The LP relaxation of problem (2.3) can be written as follows:

$$
\begin{aligned}
&-\sum_{ij} p_{ij} r_{ij} + \min \sum_{i} \sum_{j} (p_{ji} - p_{ij})\pi_i \\
&\pi_j-\pi_i \geq r_{ij} && \forall i,j \in U \text{ s. t. } p_{ij}>0, \qquad (2.4) \\
&\pi_i \geq a_i && \forall i \in U, \\
&-\pi_i \geq -b_i && \forall i \in U.
\end{aligned}
$$

Let us call $t_i = \sum_j (p_{ji} - p_{ij})$, $\forall i \in U$, $K=-\sum_{ij} p_{ij} r_{ij}$, and introduce a dummy variable π_s, then (2.4) can be written equivalently as follows:

$$
\begin{aligned}
&K + \min \sum_{i} t_i \pi_i \\
&\pi_j-\pi_i \geq r_{ij} && \forall i,j \in U \text{ s. t. } p_{ij}>0 \qquad (2.5) \\
&\pi_i - \pi_s \geq a_i && \forall i \in U \\
&\pi_s - \pi_i \geq -b_i && \forall i \in U.
\end{aligned}
$$

Problem (2.5) is a potential problem on a network $G'=(N,A')$, where the node set N is given by $U \cup \{s\}$ and the arc set $A' = \{(i,j): i,j \in U,\ p_{ij}>0\} \cup \{(s,i): i \in U\} \cup \{(i,s): i \in U\}$. The constraints matrix of problem (2.5) is totally unimodular. Due to integrality of coefficients r_{ij}, a_i and b_i, and to totally unimodularity, any optimal basic solution π^* of problem (2.5) is integer and gives the optimal solution of (2.3); consequently, it gives also the solution to the original QSAP, if the value of the optimal solution is less than M. Solution π^* can be found in polynomial time by means of linear programming. Hence we can state the following theorem.

Theorem 2.1

Any QSAP defined on a convex bipartite graph with cost given by (2.2) can be solved in polynomial time, if the value of the optimal solution is less than M.

Note that if the QSAP has a solution which includes some penalties the corresponding potential problem (2.5) is infeasible.

Problem (2.5) can be solved very efficiently by means of a standard network flow algorithm [1]. In fact the dual of (2.5) is

$$\max \sum_{ij\in U} r_{ij} y_{ij} + \sum_{i\in U} a_i y_{si} - \sum_{i\in U} b_i y_{it}$$

$$Ey = t, \tag{2.6}$$

$$y \geq 0,$$

where E is the node/arc incidence matrix relative to graph G', arcs do not have capacities, the cost of one arc $(i,j)\in A'$ is given by r_{ij}, if $i,j \in U$, a_j if $i=s$, and $-b_i$ if $j=s$, and the demand of flow at node i is equal to t_i if $i\in U$, and zero if $i=s$.

3. Conclusions

In the present note we showed that a particular class of QSAP's defined on convex bipartite graphs can be solved in polynomial time. The idea is based on a simple transformation into a network flow problem whose size is a linear function of the size of the original QSAP. This class of problems has relevant application in practical cases as the SSP and some other transportation problems. As the usual size of those problem is quite large, it is important to have very efficient methods to find the optimal solution.

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