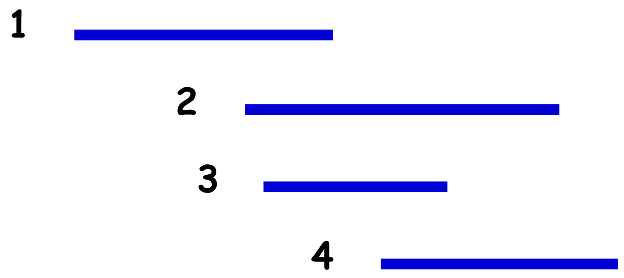
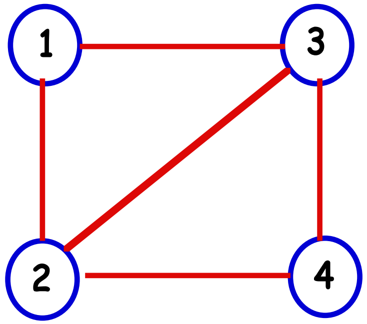
SHIFTABLE INTERVALS

Federico Malucelli (Politecnico di Milano)
Sara Nicoloso (IASI - CNR Roma)

INTERVAL GRAPHS



model

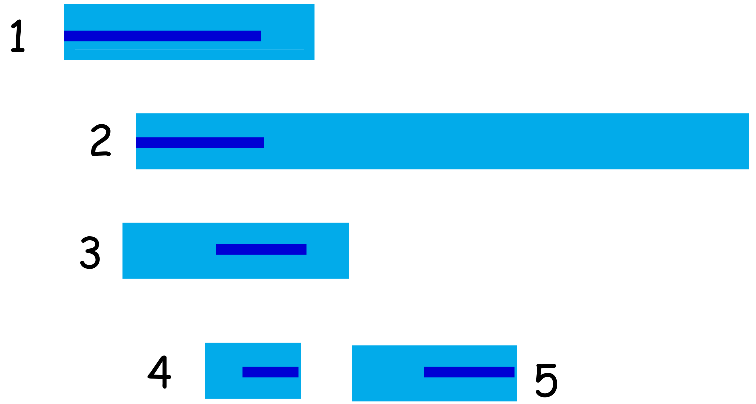


intersection graph

Applications:
Biology, Chemistry, Archeology, Scheduling,…

Easy problems on IG:
Maximum independent set (Minimum clique cover), Maximum clique (Minimun coloring), Minimum dominating set

SHIFTABLE INTERVALS



Set of triples $T = \{t_i = < l_i, r_i, \lambda_i >, i=1,\ldots,n\}$

$[l_i, r_i]$: window i

λ_i : length of interval i

we assume that l_i, r_i, λ_i are non negative integers an that $r_i - l_i \geq \lambda_i$

Intervals are allowed to shift inside the window

The difference between the left endpoint of the i^{th} interval and l_i is called placement φ_i

Different placements yield different models and interval graphs

All feasible placements define a family of Interval Graphs F_T

PROBLEMS DEFINED ON SHIFTABLE INTERVALS

Problem Min $f(T)$ [Max $f(T)$, respectively]

Given : set of triples T, and
a function f: $F_T \rightarrow Z_+$

Find : a graph $G \in F_T$ (i.e. a placement)
such that $f(G)$ is minimum [maximum, respectively]

Examples:

Min clique number,
Max stability number,
Min size of the dominating set,
Max clique number, …

OVERVIEW

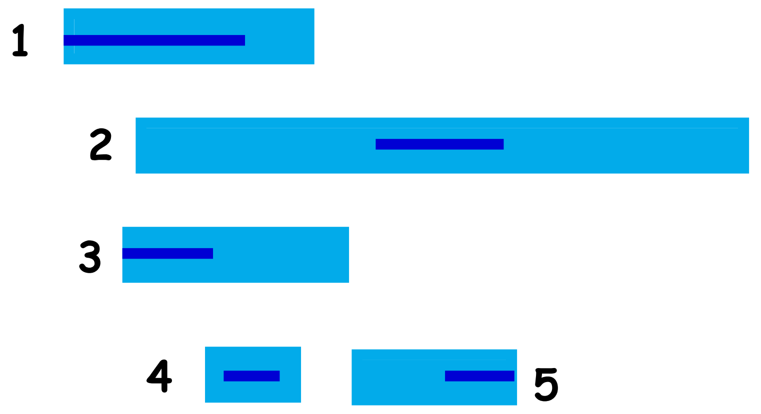
- **Definitions and properties of F_T**

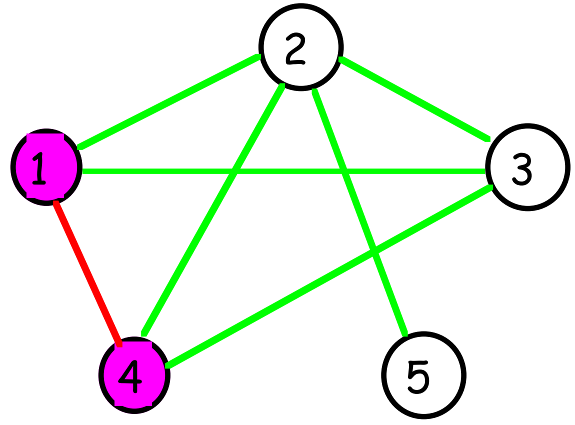
- **Minimum maximum clique (Min $\omega(T)$):**
  complexity, bounds, special easy cases

- **Maximum maximum independent set (Max $\alpha(T)$):**
  complexity, bounds, special easy cases

- **Minimum dominating set size (Min $d(T)$):**
  complexity, bounds, special easy cases

PROPERTIES OF F_T





H_T

H_T: Interval Graph defined by the windows

strong edge: it belongs to any $G \in F_T$ (whatever are the placements, the two intervals intersect);

weak edges: all the edges that are not strong

overlapping vertex: whatever is the placement of the corresponding interval a portion of the window is always "covered".

PROPERTIES OF F_T

Theorem:

There are no strong edges between non-overlapping vertices.

Theorem:

Consider a non-overlapping vertex *v*: all the vertices conneted to *v* by strong edges are overlapping and pairwise connected by strong edges.

A triple set *T* is proper if Interval Graph H_T is a proper interval graph.

(there is no interval properly contained into any other)

MIN MAX CLIQUE (Min $\omega(T)$)

All the results apply also to the Min Max chromatic number.

Theorem:

The problem is NP-Hard

reduction from 3-Partition:

given $3m+1$ positive integers $b_1,\ldots, b_{3m}, B$

$\Sigma\, b_i = mB$

find a partition into m subsets of 3 elements whose sum is B

$T = \{\ t_i = \langle 0,\ B,\ b_i \rangle,\ i=1,\ldots,3m \}$

Does there exist a placement of the intervals such that the clique number of the corresponding Interval Graph is no more than m?

MIN MAX CLIQUE

Lower and Upper bounds

Let $A = (V, E^S)$ be the subgraph of H_T defined by the strong edges.

Theorem:

$\omega(A) \leq \text{Min } \omega(T) \leq \omega(H_T)$

A is a subgraph of any $G \in F_T$;
any $G \in F_T$ is a subgraph of H_T.

$\omega(A)$ can be computed in polynomial time even if A is not an interval graph

Any complete subgraph of A contains at most one non overlapping vertex

$$\omega(A)=\{\omega(A'), \max\{|\{v\} \cup Adj^S(v)|\}: v \in V^N\}$$

where A' is the subgraph of A induced by the overlapping vertices (which is an Interval Graph)

A max clique in A is either composed by
one non overlapping vertex adjacent to overlapping ones,
or by overlapping vertices only.

Consider the Min $\omega(T)$ problem in decisional form

Let $c_h(W)$ be the rightest coordinate contained into exactly h intervals of W (0 if such coordinate does not exist).

```
Algorithm mM-C;
begin
for i = 1,…,n do φi := undefined;
let W := ø;
for i = 1,…,n do begin
        if ch(W) > ri – λi then return(NO) and stop
        else    begin
                φi = max {0, ch(W) – li}; W := W ∪ {(ti,φi)};
                end;
        end;
return(YES);
end.
```

Algorithm mM-*C* solves exactly the Min $\omega(T)$ problem in decision form when *T* is a set of proper non increasing triples.

A triple set *T* is *proper non increasing* if and only if H_T is proper and $\lambda_i \geq \lambda_j$ for any $i < j$.

The optimization problem can be solved by applying a binary search on *h*.

The complexity is $O(n \log n)$.

MAX MAX CLIQUE (Max $\omega(T)$)

The problem can be solved trivially:

Find the maximum clique C of H_T;
Place the intervals so that those whose window belongs to C intersect.

MIN MAX INDEPENDENT SET (Max $\alpha(T)$)

All the results apply also to the Max Max Clique Cover.

Theorem:

The problem is NP-Hard

Reduction from the problem of minimizing the tardy jobs in a one machine scheduling with ready and due times

ready time $\leftrightarrow$ l_i

due times $\leftrightarrow$ r_i

processing times $\leftrightarrow$ λ_i

jobs scheduled on time $\leftrightarrow$ **independent set**

MAX MAX INDEPENDENT SET (Max $\alpha(T)$)

Lower and Upper bounds

Let $A = (V, E^S)$ be the subgraph of H_T defined by the strong edges.

Theorem:

$\alpha(H_T) \leq \text{Max } \alpha(T) \leq \alpha(A)$

A is a subgraph of any $G \in F_T$;
any $G \in F_T$ is a subgraph of H_T.

$\alpha(A)$ can be computed easily as in the case of $\omega(A)$

```
Algorithm MM-IS
begin
    for i = 1,…,n do    φi := 0;
    let Y := ø;
    p := l1;
    for i = 1,…,n do
        if p ≤ ri – λi      then begin
                                φi := max {0,p – li};
                                Y := Y ∪ {<ti,φi>};
                                p := li + φi + λi
                            end
end.
```

Algorithm mM-*C* solves exactly the Max $\alpha(T)$ problem when T is a set of proper non decreasing triples.

A triple set T is *proper non-decreasing* if and only if H_T is proper and $\lambda_i \leq \lambda_j$ for any $i < j$.

If the interval are already sorted the complexity of the algorithm is linear.

MIN MAX INDEPENDENT SET (Min $\alpha(T)$)

The problem can be solved trivially

by exploiting the equivalence between the maximum independent set problem and the minimum covering by cliques in interval graphs

Find the minimum covering by cliques Γ of H_T;
For each clique C in Γ place the intervals so that those whose window belongs to C intersect.

MIN DOMINATING SET (Min $d(T)$)

Given a graph $G = (V,E)$ a subset of nodes $D_G \subseteq V$ is a *dominating set* iff for any $u \in V \setminus D_G$ there exists a $v \in D_G$ such that edge $(u,v) \in E$.

Theorem:

The problem of finding the placement that minimizes the cardinality of the dominating set (Min $d(T)$) is NP-Hard.

Reduction from 3-Partition.

MIN DOMINATING SET (Min $d(T)$)

Lower and Upper bounds

Theorem:

$|D_H| \leq d(T) \leq \alpha(H).$

The *derived set of triples* T_d associated to T is obtained by removing from T all triples whose window properly contains another window of T.

H_{T_d} is a proper Interval Graph

Theorem:

$d(T) \leq d(T_d).$

$d(T_d)$ can be computed in polynomial time by a greedy algorithm

Algorithm Greedy:
let $D := \emptyset$;
unmark all triples in T;
repeat
    let π be the leftmost right endpoint of an unmarked triple in $T \setminus D$
    let $A = \{i\colon l_i \le \pi \le r_i,\ i \notin D\}$;
    set $\varphi_i = \min\{r_i - l_i - \lambda_i,\ \pi - l_i\}$ for all $i \in A$;
    SELECTION PHASE:
    let $j \in A$ be such that $l_j + \varphi_j + \lambda_j = \max\{l_i + \varphi_i + \lambda_j : i \in A\}$;
    insert j into D;
    mark all triples t_i such that $l_i \le l_j + \varphi_j + \lambda_j$;
until all triples are marked.

The complexity of the algorithm is $O(n^2)$

CONCLUSIONS

- Extension of the Interval Graphs
- New modeling framework
- Characterization of classical problems (complexity and easy cases)

- Computational experience in a companion paper

FUTURE WORKS

- Characterization of the family of graphs F_T
- Extension to Circular Arc Graphs